Fuzzy Mathematics - II

* Unit - I

Projections & cylindrical extensions, binary fuzzy relations on singal set. Fuzzy equivalence relations, fuzzy compatibility relations, fuzzy ordering relations, fuzzy morphisms, suprimum, i-composition & infimum ωi composition

* Unit - II

fuzzy relation eqns, problem partiting, soln methods, fuzzy relational eqn based on suprimum i & infimum ωi compositions, approximate solutions.

* Unit - III

fuzzy prapositions, fuzzy contifiers, lingu hedges, inference from conditional fuzzy qualified & quantified prapositions.

* Unit - IV

fuzzy Algebra, fuzzy groups & fuzzy their basic properties.

* Unit - V

Examples, seminors, group discussion, on above four units.

* Recommanded books

1) George J klir. Bo Yuan, Fuzzy sets & fuzzy logic Theory & applications, PHI ltd (2000)

2) John Mordeson, Fuzzy Mathematics, springer, 2001.

Unit - I

Fuzzy Relation

let $x_1, x_2, \ldots, x_n$ be n number of crisp sets. a relation (crisp) is a subset of $X_1 \times X_2 \times X_3 \times \ldots \times X_n = \overset{n}{\underset{i=1}{X}} x_i$

we denote this relation by $R(x_1, x_2, \ldots, x_n)$

A fuzzy relation R on $x_1, x_2, \ldots, x_n$ is a fuzzy set defined on the cartesion product $X_1 \times X_2 \times \ldots \times X_n$.
where each n touple ~~baren~~ $(x_1, x_2, \ldots x_n)$ have a degree of relationship between 0 & 1. The value one indicates complete relationship between the elements & the value zero indicates no relationship between the elements

ex.

1) let R be fuzzy relation between two sets $X = \{SAN, KOL\}$, $Y = \{SOL, SAT, PUNE\}$ which represent the relation concept 'very'. The relation R can be represented by the following fuzzy matrix.

$$R = \begin{array}{c|ccc} & SOL & SAT & PUNE \\ \hline SAN & 0.64 & 0.48 & 0.9 \\ KOL & 0.84 & 0.48 & 1 \end{array}$$

• Remark :-

If R is a fuzzy relation between x_1 & X_2 $R : x_1 \times X_2 \longrightarrow I$, then if X_1 & X_2 are finite matrix whose elements are the relationship values in the interval I.

* Projection & Cylindrical Extension

Projection

let $\mathbb{N}_n = \{1, 2, \dots, n\}$

let $\bar{x} = \{ \langle x_i \rangle \mid i \in \mathbb{N}_n \}$

$\bar{y} = \{ \langle y_j \rangle \mid j \in J \}$, $J \subseteq \mathbb{N}$

we see that $\bar{y}$ is subset of $\bar{x}$ iff $y_j = x_j \quad \forall j \in J$

If $\bar{y}$ is subsequence of $\bar{x}$ then we write $\bar{y} < \bar{x}$

let $R(X_1, X_2, \dots, X_n)$ be a fuzzy relation on the family, $X = \{X_1, X_2, \dots, X_n\}$

let Y be a subfamily of X.

The projection of R on Y is defined by

$$\boxed{(R \downarrow Y)(\bar{y}) = \max_{\bar{y} < \bar{x}} R(\bar{x})}$$

Similarly, if R is fuzzy relation on the subfamily Y then it's cylindrical extension is defined by,

$$[R \uparrow X - Y](\bar{x}) = R(\bar{y})$$

where $\bar{y} < \bar{x}$

- Examples

1) $X_1 = \{0,1\}$, $X_2 = \{0,1\}$, $X_3 = \{0,1,2\}$ & $R(X_1, X_2, X_3)$ be a fuzzy relation given by

$$R_{123} = \frac{0.4}{0,0,0} + \frac{0.9}{0,0,1} + \frac{0.2}{0,0,2} + \frac{1}{0,1,0} + \frac{0}{0,1,1} +$$

$$\frac{0.8}{0,1,2} + \frac{0.5}{1,0,0} + \frac{0.3}{1,0,1} + \frac{0.1}{1,0,2} + \frac{0}{1,1,0} + \frac{0.5}{1,1,1} + \frac{1}{1,1,2}$$

obtain the projection of R_{123} on R_{12}, R_{13}, R_{23}, R_1, R_2, R_3.

<u>Solution</u> :-

1) $R_{23} : X_2 \times X_3 \longrightarrow I$

$$R_{23}(x_2, x_3) = \max_{x_1} R_{123}(x_1, x_2, x_3)$$

$$R_{23}(0,0) = \max_{x_1} R_{123}(x_1, 0, 0)$$

$$= \max_{x_1} \{R_{123}(0,0,0), R_{123}(1,0,0)\}$$

$$= \max\{0.4, 0.5\}$$

$$= 0.5$$

$$R_{23}(0,1) = \max_{x_1} R_{123}\{x_1, 0, 1\}$$

$$= \max_{x_1} \{R_{123}(0,0,1), R_{123}(1,0,1)\}$$

$$= \max\{0.9, 0.3\}$$

$$= 0.9$$

$$R_{23}(0,2) = \max_{x_1} R_{123}\{x_1, 0, 2\}$$

$$= \max_{x_1} \{R_{123}(0,0,2), R_{123}(1,0,2)\}$$

$$= \max\{0.2, 0.1\}$$

$$= 0.2$$

$$R_{23}(1,0) = \max_{x_1} R_{123}\{x_1, 1, 0\}$$

$$= \max_{x_1} \{R_{123}(0,1,0), R_{123}(1,1,0)\}$$

$$= \max\{1, 0\}$$

$$= 1$$

$R_{23}(1,1) = \max_{x_1} (x_1, 1, 1)$

$= \max_{x_1} \{ R_{123}(0,1,1), R_{123}(1,1,1) \}$

$= \max \{ 0, 0.5 \}$

$= 0.5$

$R_{23}(1,2) = \max_{x_1} \{ x_1, 1, 2 \}$

$= \max_{x_1} \{ R_{123}(0,1,2), R_{123}(\phi,1,2) \}$

$= \max \{ 0.8, 1 \}$

$= 1$

$$R_{23} = \frac{0.5}{0,0} + \frac{0.9}{0,1} + \frac{0.2}{0,2} + \frac{1}{1,0} + \frac{0.5}{1,1} + \frac{1}{1,2}$$

$R_{12} : X_1 \times X_2 \longrightarrow I$

$R_{12}(x_1, x_2) = \max_{x_3} R_{123}(x_1, x_2, x_3)$

$R_{12}(0,0) = \max_{x_3} R_{123} \{0, 0, x_3\}$

$= \max_{x_3} \{ R_{123}(0,0,0), R_{123}(0,0,1), R_{123}(0,0,2)$

$= 0.4 \vee 0.9 \vee 0.2$

$= 0.9$

$R_{12}(0,1) = \max_{x_3} R_{123} \{ \underbrace{x_1, \cancel{x_2}}_{0,1}, x_3 \}$

$= \max_{x_3} \{ R_{123}(0,1,0), R_{123}(0,1,1), R_{13}(01,2$

$= 1 \vee 0 \vee 0.8$

$= 1$

$R_{12}(1,0) = \max_{x_3} (1, 0, x_3)$

$= \max_{x_3} (R_{123}(1,0,0), R_{123}(1,0,1), R_{123}(1,0,2)\}$

$= 0.5 \vee 0.3 \vee 0.1$

$= 0.5$

$R_{12}(1,1) = \max_{x_3} (1, 1, x_3)$

$= \max_{x_3} (R_{123}(1,1,0) \; R_{123}(1,1,1) \; R_{123}(1,1,2)\}$

$= 0 \vee 0.5 \vee 1$

$= 1$

$\therefore R_{12} = \frac{0.9}{0,0} + \frac{1}{0,1} + \frac{0.5}{1,0} + \frac{1}{1,1}$

$R_{13} : X_1 \times X_3 \longrightarrow I$

$R_{13}(x_1, x_3) = \max_{x_2} R_{123}(x_1, x_2, x_3)$

$R_{13}(0,0) = \max_{x_2} R_{123}(0, x_2, 0)$

$= \max_{x_2} (R_{123}(0,0,0) \; R_{123}(0,1,0)\}$

$= 0.4 \vee 1$

$= 1$

$R_{13}(0,1) = \max R_{123}(0, x_2, 1)$

$= \max_{x_2} (R_{123}(0,0,1), R_{123}(0,1,1))$

$= 0.9 \vee 0$

$= 0.9$

$R_{13}(0,2) = \max_{x_2} R_{123}(0, x_2, 2)$

$= \max_{x_2} \{R_{123}(0,0,2), R_{123}(0,1,2)\}$

$= 0.2 \vee 0.8$

$= 0.8$

$$R_{13}(1,0) = \max_{x_2} R_{123}(1, x_2, 0)$$

$$= \max_{x_2} \{ R_{123}(1,0,0), R_{123}(1,1,0) \}$$

$$= 0.5 \vee 0$$

$$= 0.5$$

$$R_{13}(1,1) = \max_{x_2} R_{123}(1, x_2, 1)$$

$$= \max_{x_2} \{ R_{123}(1,0,1), R_{123}(1,1,1) \}$$

$$= 0.3 \vee 0.5$$

$$= 0.5$$

$$R_{13}(1,2) = \max_{x_2} R_{123}(1, x_2, 2)$$

$$= \max_{x_2} \{ R_{123}(1,0,2), R_{123}(1,1,2) \}$$

$$= 0.1 \vee 1$$

$$= 1$$

$$R_{13} = \frac{1}{0,0} + \frac{0.9}{0,1} + \frac{0.8}{0,2} + \frac{0.5}{1,0} + \frac{0.5}{1,1} + \frac{1}{1,2}$$

$$R_1 : X_1 \longrightarrow I$$

$$R_1(0) = \max R_{123}(x_1, x_2, x_3)$$

$$= \max \{ R_{123}(0, x_2, x_3), (1, x_2, x_3) \}$$

$$= \max [R_{123}(0, x_2, x_3)]$$

$$= R_{123}(0,0,0) \vee R_{123}(0,0,1) \vee R_{123}(0,0,2) \vee R_{123}(0,1,0) \vee R_{123}(0,1,1) \vee R_{123}(0,1,2)$$

$$= 0.4 \vee 0.9 \vee 0.2 \vee 1 \vee 0 \vee 0.8$$

$$= 1$$

$$R_1(1) = \max R_{123}(x_1, x_2, x_3)$$

$$= \max \{ R_{123}(0, x_2, x_3), (1, x_2, x_3) \}$$

$$= R_{123}(1,0,0) \vee R_{123}(1,0,1) \vee R_{123}(1,0,2) \vee R_{123}(1,1,0) \vee R_{123}(1,1,1) \vee R_{123}(1,1,2)$$

$= 0.5 \vee 0.3 \vee 0.1 \vee 0 \vee 0.5 \vee 1$

$= 1$

$R_1 = \frac{1}{0} + \frac{1}{1}$

$R_2 : X_2 \rightarrow I$

$R_2(0) = \max R_{123}(x_1, x_2, x_3)$

$= \max \{ R_{123}(x_1, 0, x_3)$

$= \cancel{\max} \{$

$= R_{123}(0,0,0) \vee R_{123}(0,0,1) \vee R_{123}(0,0,2) \vee$
$R_{123}(1,0,0) \vee R_{123}(1,0,1) \vee R_{123}(1,0,2)$

$= 0.4 \vee 0.9 \vee 0.2 \vee 0.5 \vee 0.3 \vee 0.1$

$= 0.9$

$R_2(1) = \max R_{123}(x_1, x_2, x_3)$

$= \max R_{123}(x_1, 1, x_3)$

$= R_{123}(0,1,0) \vee R_{123}(0,1,1) \vee R_{123}(0,1,2) \vee$
$R_{123}(1,1,0) \vee R_{123}(1,1,1) \vee R_{123}(1,1,2)$

$= 1 \vee 0 \vee 0.8 \vee 0 \vee 0.5 \vee 1$

$= 1$

$R_2 = \frac{0.9}{0} + \frac{1}{1}$

$R_3 : X_3 \rightarrow I$

$R_3(0) = \max R_{123}(x_1 x_2 x_3)$

$= \max R_{123}(x_1 x_2 0)$

$= R_{123}(0,0,0) \vee R_{123}(0,1,0) \vee R_{123}(1,0,0) \vee R_{123}(1,1,0)$

$= 0.4 \vee 1 \vee 0.5 \vee 0$

$= \cancel{0.5}\ 1$

$R_3(1) = \max R_{123}(x_1 x_2, 1)$

$= R_{123}(0,0,1) \vee R_{123}(0,1,1) \vee R_{123}(1,0,1) \vee R_{123}(1,1,1)$

$= 0.9 \vee 0 \vee 0.3 \vee 0.5$

$= 0.9$

$R_3(2) = \max R_{123}(x_1, x_2, x_3)$

$= \max R_{123}(x_1, x_2, 2)$

$= R_{123}(0,0,2) \vee R_{123}(0,1,2) \vee R_{123}(1,0,2) \vee R_{123}(1,1,2)$

$= 0.2 \vee 0.8 \vee 0.1 \vee 1$

$= 1$

$$R_3 = \frac{\cancel{1}}{0} + \frac{0.9}{1} + \frac{1}{2}$$

Q) let $x_1 = \{0,1\}$, $x_2 = \{0,1\}$, $x_3 = \{0,1,2\}$

let $R(x_1, x_2) = R_{12} = \frac{0.4}{0,0} + \frac{0.9}{0,1} + \frac{0.5}{1,0} + \frac{1}{1,1}$

Explain R_{12} to R_{123} using cylindrical extension method

⇒ The cylindrical extension of R_{12} to R_{123} is given by

$R_{123}(x_1, x_2, x_3) = R_{12}(x_1, x_2)$

$R_{123} : X_1 \times X_2 \times X_3 \longrightarrow I$

$X_1 \times X_2 \times X_3 = \{0,1\} \times \{0,1\} \times \{0,1,2\}$

$= \{(0,0,0), (0,0,1), (0,0,2), (0,1,0), (0,1,1), (0,1,2), (1,0,0), (1,0,1), (1,0,2), (1,1,0), (1,1,1), (1,1,2)\}$

$R_{123}(0,0,0) = R_{12}(0,0) = 0.4$, $R_{123}(1,0,0) = R_{12}(1,0) = 0.5$

$R_{123}(0,0,1) = R_{12}(0,0) = 0.4$, $R_{123}(1,0,1) = R_{12}(1,0) = 0.5$

$R_{123}(0,0,2) = R_{12}(0,0) = 0.4$, $R_{123}(1,0,2) = R_{12}(1,0) = 0.5$

$R_{123}(0,1,0) = R_{12}(0,1) = 0.9$, $R_{123}(1,1,0) = R_{12}(1,1) = 1$

$R_{123}(0,1,1) = R_{12}(0,1) = 0.9$, $R_{123}(1,1,1) = R_{12}(1,1) = 1$

$R_{123}(0,1,2) = R_{12}(0,1) = 0.9$, $R_{123}(1,1,2) = R_{12}(1,1) = 1$

(a,s,i) (a,s,j) (a,t,i) (a,t,j)
(b,s,i) (b,s,j) (b,t,i) (b,t,j)
(c,s,i) (c,s,j) (c,t,i) (c,t,j)

$$R_{123} = \frac{0.4}{0,0,0} + \frac{0.4}{0,0,1} + \frac{0.4}{0,0,2} + \frac{0.9}{0,1,0} + \frac{0.9}{0,1,1} + \frac{0.9}{0,1,2} + \frac{0.5}{1,0,0} + \frac{0.5}{1,0,1} + \frac{0.5}{1,0,2} + \frac{1}{1,1,0} + \frac{1}{1,1,1} + \frac{1}{1,1,2}$$

3) let $X_1 = \{a, b, c\}$, $X_2 = \{s, t\}$, $X_3 = \{x, y\}$, $X_4 = \{i, j\}$

let $R(X_1, X_2, X_3, X_4) = R_{1234}$

& $$R_{1234} = \frac{0.6}{a,s,x,i} + \frac{0.4}{b,t,y,i} + \frac{0.9}{b,s,y,i} + \frac{1}{b,s,y,j} + \frac{0.6}{a,t,y,j} + \frac{0.2}{c,s,y,i}$$

compute the projections R_{124}, R_{13} & R_4

also compute the cylindrical extension

$R_{124} \uparrow X_3$, $R_{13} \uparrow \{X_2, X_4\}$, $R_4 \uparrow \{X_1, X_2, X_3\}$

$\Rightarrow$ $R_{124} : X_1 \times X_2 \times X_4 \longrightarrow I$

$$R_{124}(x_1, x_2, x_4) = \max_{x_3} R_{1234}(x_1, x_2, x_3, x_4)$$

~~$X_1 \times X_2 \times X_4 = \{ (a,s,i\ (a,b,c)\ (s,t)\ (x,y), (i,j)\}$~~

~~$= \{ (a,s,i$~~

$$R_{124}(a,s,i) = \max_{x_3} \{ (a,s,x,i), (a,s,y,i) \}$$

$= \max\{0.6, 0\}$

$= 0.6$

$$R_{124}(a,s,j) = \max_{x_3} \{ (a,s,x,j), (a,s,y,j) \}$$

$= \max\{0, 0\}$

$= 0$

$R_{124}(a,t,i) = \max\limits_{x_3} \{(a,t,x,i)\ (a,t,y,j)\}$

$= \max\{0,0\}$

$= 0$

$R_{124}(a,t,j) = \max\limits_{x_3} \{(a,t,x,j)\ (a,t,y,j)\}$

$= \max\{0, 0.6\}$

$= 0.6$

$R_{124}(b,s,i) = \max\limits_{x_3} \{(b,s,x,i),\ (b,s,y,i)\}$

$= \max\{0, 0.9\}$

$= 0.9$

$R_{124}(b,s,j) = \max\limits_{x_3} \{(b,s,x,j)\ (b,s,y,j)\}$

$= \max\{0,1\}$

$= 1$

$R_{124}(b,t,i) = \max\limits_{x_3} \{(b,t,x,i)\ (b,t,y,i)\}$

$= \max\{0, 0.4\}$

$= 0.4$

$R_{124}(b,t,j) = \max\limits_{x_3} \{(b,t,x,j)\ (b,t,y,j))$

$= \max(0,0)$

$= 0$

$R_{124}(c,s,i) = \max\limits_{x_3} \{(c,s,x,i)\ (c,x,y,i))$

$= \max(0, 0.2)$

$= 0.2$

$R_{123}(c,s,j) = \max_{x_3} \{(c,s,x,j)\ (c,s,y,j)\}$

$= \max(0,0)$

$= 0$

$R_{123}(c,t,i) = \max_{x_3} \{(c,t,x,i), (c,t,y,i)\}$

$= \max(0,0)$

$= 0$

$R_{123}(c,t,j) = \max_{x_3} \{(c,t,x,j), (c,t,y,j)\}$

$= \max(0,0)$

$= 0.$

$$R_{123} = \frac{0.6}{a,s,i} + \frac{0}{a,s,j} + \frac{0}{a,t,i} + \frac{0.6}{a,t,j} + \frac{0.9}{b,s,i} + \frac{1}{b,s,j} + \frac{0.4}{b,t,i} + \frac{0}{b,t,j} + \frac{0.2}{c,s,i} + \frac{0}{c,s,j} + \frac{0}{c,t,i} + \frac{0}{c,t,j}$$

$$R_{124} = \frac{0.6}{a,s,i} + \frac{0.6}{a,t,j} + \frac{0.9}{b,s,i} + \frac{1}{b,s,j} + \frac{0.4}{b,t,i} + \frac{0.2}{c,s,i}$$

~~$R_{13} : X_1 \times X_3 \longrightarrow I$~~

~~$R_{13}(x_1, x_3) = \max R_{1234}(x_1, x_2, x_3, x_4)$~~

$R_{1234}(x_1, x_2, x_3, x_4) = R_{124}(\cancel{x_1, x_2}$

~~$R_{124} = \frac{0.6}{a,s,i}$~~

$X_1 \times X_2 \times X_3 \times X_4 = \{(a,s,x,i)\ (a,s,x,j)\ (a,s,y,i)$
$(a,s,y,j)\ \cancel{(a,s,x,i)}\ (a,t,x,i)\ (a,t,x,j)\ (a,t,y,i)$
$(a,t,y,j)\ (b,s,x,i)\ (b,s,x,j)\ (b,s,y,i)\ (b,s,y,j)$
$(b,t,x,i)\ (b,t,x,j)\ (b,t,y,i)\ (b,t,y,j)\ (c,s,x,i)$
$(c,s,x,j)\ (c,s,y,i)\ (c,s,y,j)\ (c,t,x,i)\ (c,t,x,j)\ (c,t,y,i)\ (c,t,y,j)$

$$R_{124} \uparrow X_3 = R_{1234} = \frac{0.6}{a,s,x,i} + \frac{0.6}{a,s,y,i} + \frac{0.9}{b,s,y,i} + \frac{1}{b,s,y,j} + \frac{0.4}{b,t,y,i} + \frac{0.2}{c,x,y,i}$$

$$R_{13}(a,x) = \max \{ \{ (a,s,x,i), (a,s,x,j), (a,t,y,i)\ (a,t,y,j) \}$$
$$= \max \{ 0.6, 0.6 \}$$
$$= 0.6$$

$R_4 : X_4 \longrightarrow I$

$$R_4(i) = \max \{ (a,s,x,i), (a,s,y,i), (a,t,x,i)\ (a,t,y,i)\ (b,s,x,i)\ (b,s,y,i)\ (b,t,x,i)\ (b,t,y,i)\ (c,s,x,i)\ (c,s,y,i), (c,t,x,i)\ (c,t,y,i) \}$$
$$= \max \{ 0.6, 0.4, 0.9, 0.2 \}$$
$$= 0.9$$

$$R_4(j) = \max \{ (a,s,x,j), (a,s,y,j), (a,t,x,j)\ (a,t,y,j)\ (b,s,x,j)\ (b,s,y,i), (b,t,x,j)\ (b,t,y,i)\ (c,s,x,i)\ (c,s,y,i), (c,t,x,i)\ (c,t,y,i) \}$$
$$= \max \{ 0.6, 1 \}$$
$$= 1$$

$$R_4 = \frac{0.9}{i} + \frac{1}{j}$$

$$R_{1234}(x_1,x_2,x_3,x_4) = R_4(x_4)$$

$$R_4 \uparrow \{x_1, x_2, x_3\} = R_{1234} = \frac{0.9}{a,s,x,i} + \frac{1}{a,s,x,j} + \frac{0.9}{a,s,y,i} + \frac{1}{a,s,y,j}$$

$$+ \frac{0.9}{a,t,x,i} + \frac{1}{a,t,x,j} + \frac{0.9}{a,t,y,i} + \frac{1}{a,t,y,j} + \frac{0.9}{b,s,x,i} + \frac{1}{b,s,x,j} +$$

$+ \frac{0.9}{b,s,y,i} + \frac{1}{b,s,y,j} + \frac{0.9}{b,t,x,i} + \frac{1}{b,t,x,j} + \frac{0.9}{b,t,y,i} + \frac{1}{b,t,y,j} + \frac{0.9}{c,s,x,i}$

$+ \frac{1}{c,s,x,j} + \frac{0.9}{c,s,y,i} + \frac{1}{c,s,y,j} + \frac{0.9}{c,t,x,i} + \frac{1}{c,s,x,j} + \frac{0.9}{c,t,y,i} + \frac{1}{c,t,y,j}$

$R_{13} : X_1 \times X_3 \longrightarrow I$

$R_{13}(x_1, x_3) = \max_{x_2, x_4} R_{1234}(x_1, x_2, x_3, x_4)$

$X_1 \times X_3 = \{(a,b,c), (x,y)\}$

$= \{(a,x) (a,y) (b,x) (b,y) (c,x), (c,y)\}$

$R_{13}(a,x) = \max_{x_2, x_4} \{(a,s,x,i), (a,s,x,j),$
$(a,t,x,i), (a,t,x,j)\}$

$= \max\{0.6, 0, 0, 0\}$

$= 0.6$

$R_{13}(a,y) = \max_{x_2, x_4} \{(a,s,y,i), (a,s,y,j), (a,t,y,i)$
$(a,t,y,j)\}$

$= \max\{0, 0, 0, 0.6\}$

$= 0.6$

$R_{13}(b,x) = \max\{(b,s,x,i) (b,s,x,j) (b,t,x,i)$
$(b,t,x,j)\}$

$= \max\{0, 0, 0, 0\}$

$= 0$

$R_{13}(b,y) = \max\{(b,s,y,i) (b,s,y,j), (b,t,y,i),$
$(b,t,y,j)\}$

$= \max\{0.9, 1, 0.4, 0\}$

$= 1$

$$R_{13}(c,x) = \max\{(c,s,x,i),(c,s,x,j),(c,t,x,i),(c,t,x,j)\}$$
$$= \max\{0,0,0,0\}$$
$$= 0$$

$$R_{13}(c,y) = \max\{(c,s,y,i),(c,s,y,j),(c,t,y,i),(c,t,y,j)\}$$
$$= \max\{0.2,0,0,0\}$$
$$= 0.2$$

$$\therefore R_{13} = \frac{0.6}{a,x} + \frac{0.6}{a,y} + \frac{1}{b,y} + \frac{0.2}{c,y}$$

$$R_{1234}(x_1,x_2,x_3,x_4) = R_{13}(x_1,x_3)$$

$$R_{13}\uparrow\{x_2,x_4\} = R_{1234} = \frac{0.6}{a,s,x,i} + \frac{0.6}{a,s,x,j} + \frac{0.6}{a,s,y,i} + \frac{0.6}{a,s,y,j} + \frac{0.6}{a,t,x,i} + \frac{0.6}{a,t,x,j} + \frac{0.6}{a,t,y,i} + \frac{0.6}{a,t,y,j} + \frac{1}{b,s,y,i} + \frac{1}{b,s,y,j} + \frac{1}{b,t,y,i} + \frac{1}{b,t,y,j} + \frac{0.2}{c,s,y,i} + \frac{0.2}{c,s,y,j} + \frac{0.2}{c,t,y,i} + \frac{0.2}{c,t,y,j}$$

* Remark

- Projection of fuzzy relation & cylindrical extension of fuzzy relation are not inverses of each other.

From the above example we have projection of R_{1234} to R_{124} with have eigen extended this R_{124} to R_{1234} & we observed that it is not similar to the original R_{1234}

* Binary fuzzy Relation

let X & Y be two sets a fuzzy relation R on $X \times Y$ is a fuzzy set defined on $X \times Y$.

i.e., $R : X \times Y \longrightarrow I$

we denote this fuzzy relation by $R(X, Y)$.

* Domain of fuzzy Relation

Domain of fuzzy relation $R(X,Y)$ is defined as, $\text{dom} R(x) = \max_{y} R(x,y) \quad \forall x \in X$

* Range of fuzzy relation

Range of the fuzzy relation $R(X,Y)$ is defined by, $\text{Ran } R(y) = \max_{x} R(x,y) \quad \forall y \in Y$

* Height of fuzzy relation:

Height of fuzzy relation $R(X,Y)$ is defined by,

$$h(R) = \max_{x,y} R(x,J) \quad \forall x, X, y \in Y$$

- Note:

Whenever X & Y are finite sets then the fuzzy relation $R(X,Y)$ can be represented by a matrix whose elements are the degree of relationship

i.e., $R = [\, r_{xy} \,]$ where, $r_{xy} = R(x,y)$

~~sum~~ sometimes the fuzzy relation are can be be represented by sagittal (arrow) diagram.

The relationship between the elements is represented by an arrow & the non-zero relationship value is return on the arrow.

Example

1) A fuzzy relation R on $X \times Y$ is given by

$$R = \frac{1}{x_1, y_1} + \frac{0.2}{x_1, y_2} + \frac{0.4}{x_2, y_3} + \frac{0.3}{x_2, y_4} + \frac{1}{x_3, y_1} + \frac{0.2}{x_3, y_2} + \frac{0.4}{x_4, y_3} + \frac{0.3}{x_4, y_4}$$

$X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$

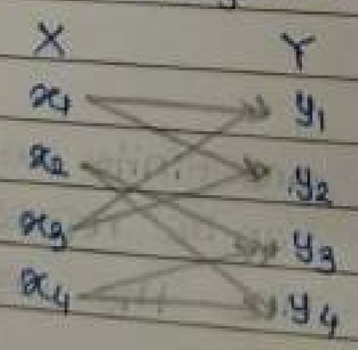
write the relation R as a matrix & also draw a sagittal diagram find dom R, Ran R, $h(R)$

$\Rightarrow$ As a matrix, the given relation are can be written as,

$$R = \begin{array}{c|cccc} & y_1 & y_2 & y_3 & y_4 \\ \hline x_1 & & & & \\ x_2 & & & & \\ x_3 & & & & \\ x_4 & & & & \end{array}$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1 & 0.2 & 0 & 0 \\ 0 & 0 & 0.4 & 0.3 \\ 1 & 0.2 & 0 & 0 \\ 0 & 0 & 0.4 & 0.3 \end{bmatrix} \end{array}$$

The sagittal diagram of R given by.



$$\text{dom } R(x) = \max_{y} R(x,y) \quad \forall x \in X$$

$$\text{dom } R(x_1) = \max_{y} R(x_1, y)$$

$$= \max\{R(x_1,y_1) R(x_1,y_2) R(x_1,y_3) R(x_1,y_4)\}$$

$$= \max\{1, 0.2, 0, 0\}$$

$$= 1$$

$$\text{dom } R(x_2) = \max_{y} R(x_2, y)$$

$$= \max\{R(x_2,y_1) R(x_2,y_2) R(x_2,y_3) R(x_2 y_4)\}$$

$$= \max\{0, 0, 0.4, 0.3\}$$

$$= 0.4$$

$$\text{dom } R(x_3) = \max_{y} R(x_3, y)$$

$$= \max\{R(x_3,y_1) R(x_3,y_2) R(x_3,y_3) R(x_3,y_4)$$

$$= \max\{1, 0.2, 0, 0\}$$

$$= 1$$

$$\text{dom } R(x_4) = \max R(x_4, y)$$

$$= \max\{R(x_4,y_1) R(x_4,y_2) R(x_4,y_3) R(x_4,y_4)$$

$$= \max\{0, 0, 0.4, 0.3\}$$

$$= 0.4$$

$$\text{dom } R = \frac{1}{x_1} + \frac{0.4}{x_2} + \frac{1}{x_3} + \frac{0.4}{x_4}$$

note that,

dom R is a fuzzy set defined on X.

$$\text{ran } R(y) = \max_{x} R(x,y)$$

$$\text{ran } R(y_1) = \max_{x} R(x, y_1)$$

$$= \max\{R(x_1,y_1), R(x_2,y_1) R(x_3,y_1) R(x_4,y_1)\}$$

$$= \max\{1, 0, 1, 0\}$$

$$= 1$$

$$\text{ran } R(y_2) = \max_x (x, y_2)$$

$$= \max_x \{R(x_1, y_2)\ R(x_2, y_2)\ R(x_3 y_2)\ R(x_4, y_2)\}$$

$$= \max\{0.2, 0, 0.2, 0\}$$

$$= 0.2$$

$$\text{ran } R(y_3) = \max_x (x, y_3)$$

$$= \max_x \{R(x_1, y_3), R(x_2, y_3), R(x_3, y_3), R(x_4, y_3)\}$$

$$= \max\{0, 0, 0.4, 0.4\}$$

$$= 0.4$$

$$\text{ran } R(y_4) = \max_x (x, y_4)$$

$$= \max\{R(x_1, y_4)\ R(x_2 y_4)\ R(x_3, y_4)\ R(x_4, y_4)\}$$

$$= \max\{0, 0, 0, 0.3\}$$

$$= 0.3$$

$$\text{ran }(R) = \frac{1}{y_1} + \frac{0.2}{y_2} + \frac{0.4}{y_3} + \frac{0.3}{y_4}$$

Note that, ran R is fuzzy is set defined on Y.

$$h(R) = \max_{(x,y)} R(x, y)$$

$$h(R) = 1$$

* Inverse ~~Ra~~ of Fuzzy Relation

If R is fuzzy relation on $X \times Y$ then inverse fuzzy relation $R^{-1}(Y, X)$ is defined by,

$R^{-1}(x, y) = R(y, x)$

e.g.

If R is a fuzzy relation on $X \times Y$ where $X = \{x_1, x_2, x_3\}$ & $Y = \{y_1, y_2\}$

$$R = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.4 \\ 0.9 & 0.1 \end{bmatrix} \quad \text{then} \quad R^{-1} = \begin{bmatrix} 1 & 0.5 & 0.9 \\ 0 & 0.4 & 0.1 \end{bmatrix}$$

The matrix of R^{-1} is the transpose of the matrix ~~of~~ for R.

Note that, $R^{-1}(y, x) = R(x, y)$

$$(R^{-1})^{-1}(y, x) = R^{-1}(x, y)$$

$$(R^{-1})^{-1}(x, y) = R^{-1}(y, x)$$

$$(R^{-1})^{-1}(x, y) = R(x, y)$$

$$(R^{-1})^{-1} = R$$

* Composition of two fuzzy relation

In $P(X, Y)$ and $Q(Y, Z)$ are two fuzzy relation then the standard composition of P & Q is a binary fuzzy relation on $X \times Z$ and is defined by ~~$(P \circ Q)(x, z)$~~

$$(P \circ Q)(x, z) = \max_{y} \min \{P(x, y), Q(y, z)\}$$

This composition is also called as max min composition

* Theorem:

Statement:

If $P(X,Y)$, $Q(Y,Z)$ & $R(Z,W)$ are binary fuzzy relations then

i) $[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X)$

ii) $[P(X,Y) \circ Q(Y,Z)] \circ R(Z,W) = P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)]$

proof

i) $(z,x) \in Z \times X$ be arbitrary then consider,

$$[P(X,Y) \circ Q(Y,Z)]^{-1}(z,x) = [P(X,Y) \circ Q(Y,Z)](x,z)$$

$$= \max_{y} \min \{P(x,y), Q(y,z)\}$$

$$= \max_{y \in Y} \min \{P^{-1}(y,x), Q^{-1}(z,y)\}$$

$$= \max_{y \in Y} \min \{Q^{-1}(z,y), P^{-1}(y,x)\}$$

$$= (Q^{-1} \circ P^{-1})(z,x)$$

$$[P(X,Y) \circ Q(Y,Z)]^{-1}_{(z,x)} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X)](z,x)$$

$$[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X)$$

ii) let $(x,w) \in X \times W$

consider,

$$[(P \circ Q) \circ R](x,w) = \max_{z \in Z} \min \{(P \circ Q)(x,z), R(z,w)\}$$

$$= \max_{z \in Z} \min \left\{ \max_{y \in Y} \min \left(P(x,y), Q(y,z) \right), R(z,w) \right\}$$

$$= \max_{z \in Z} \max_{y \in Y} \min \left\{ P(x,y), Q(y,z), R(z,w) \right\}$$

$$= \max_{\substack{z \in Z \\ y \in Y}} \left\{ P(x,y) \wedge Q(y,z) \wedge R(z,w) \right\}$$

$$= \max_{y \in Y} P(x,y) \wedge \max_{z \in Z} \left(Q(y,z) \wedge R(z,w) \right)$$

$$= \max_{y \in Y} \min \left\{ P(x,y), Q o R(y,w) \right\}$$

$$= [P o (Q o R)](x,w)$$

Hence we have,

$(PoQ) o R = P o (Q o R)$

max min composition of fuzzy relation is associative.

However, max min composition is not commutative

MCQ i.e., $PoQ \neq QoP$

* Def^n

If X, Y, Z are finite sets then the relation $P(X,Y)$ & $Q(Y,Z)$ can be written in matrix form as $P = [P_{ik}]$, $Q = [q_{kj}]$, then the composition $R = PoQ = [P_{ik}] o [q_{kj}]$

$$R = [r_{ij}]$$

where, $r_{ij} = \max_{k} \min \{ P_{ik}, q_{kj} \}$

Ex.

1) If $P(X,Y)$ & $Q(Y,Z)$ be the fuzzy relation, given by following matrices

$$P = \begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.3 \\ 1 & 0 & 0.5 & 0.3 \end{bmatrix}$$

Find out $P \circ Q$

→ The composition of P & Q are given by,

$R = P \circ Q$

$R = [r_{ij}]$

$r_{ij} = \max_k (p_{ik} \wedge q_{kj})$ $\{k = 1, 2, 3, \ldots\}$

$r_{11} = \max_k (p_{1k} \wedge q_{k1})$

$= \max_k (p_{11} \wedge q_{11}, p_{12} \wedge q_{21}, p_{13} \wedge q_{31})$

$= \max_k (0.3 \wedge 0.9, 0.5 \wedge 0.3, 0.8 \wedge 1)$

$= \max (0.3, 0.3, 0.8)$

$= 0.8$

$r_{12} = \max_k (p_{1k} \wedge q_{k2})$

$= \max (p_{11} \wedge q_{12}, p_{12} \wedge q_{22}, p_{13} \wedge q_{32})$

$= \max (0.3, 0.2, 0)$

$= 0.3$

$r_{13} = \max_k (p_{1k} \wedge q_{k3})$

$= \max (p_{11} \wedge q_{13}, p_{12} \wedge q_{23}, p_{13} \wedge q_{33})$

$= \max (0.3, 0, 0.5)$

$= 0.5$

$r_{14} = \max_k (P_{1k} \wedge q_{k4})$

$= \max(P_{11} \wedge q_{14}, P_{12} \wedge q_{24}, P_{13} \wedge q_{34})$

$= \max(0.3, 0.5, 0.5)$

$= 0.5$

$r_{21} = \max(P_{2k} \wedge q_{k1})$

$= \max(P_{21} \wedge q_{11}, P_{22} \wedge q_{21}, P_{23} \wedge q_{31})$

$= \max(0, 0.3, 1)$

$= 1$

$r_{22} = \max(P_{2k} \wedge q_{k2})$

$= \max(P_{21} \wedge q_{12}, P_{22} \wedge q_{22}, q_{23} \wedge q_{32})$

$= \max(0, 0.2, 0)$

$= 0.2$

$r_{23} = \max(P_{2k} \wedge q_{k3})$

$= \max(P_{21} \wedge q_{13}, P_{22} \wedge q_{23}, q_{23} \wedge q_{33})$

$= \max(0, 0, 0.5)$

$= 0.5$

$r_{24} = \max(P_{2k} \wedge q_{k4})$

$= \max(P_{21} \wedge q_{14}, P_{22} \wedge q_{24}, q_{23} \wedge q_{34})$

$= \max(0, 0.7, 0.5)$

$= 0.7$

$r_{31} = \max(P_{3k} \wedge q_{k1})$

$= \max(P_{31} \wedge q_{11}, P_{32} \wedge q_{21}, P_{33} \wedge q_{31})$

$= \max(0.4, 0.3, 0.5)$

$= 0.5$

$r_{32} = \max \{ p_{3k} \wedge q_{k2})$
$= \max (p_{31} \wedge q_{12}, p_{32} \wedge q_{22}, p_{33} \wedge q_{32})$
$= \max (0.4, 0.2, 0)$
$= 0.4$

$r_{33} = \max (p_{3k} \wedge q_{k3})$
$= \max (p_{31} \wedge q_{13}, p_{32} \wedge q_{323}, p_{33} \wedge q_{33})$
$= \max (0.4, 0, 0.5)$
$= 0.5$

$r_{34} = \max (p_{3k} \wedge q_{k4})$
$= \max (p_{31} \wedge q_{14}, p_{32} \wedge q_{24}, p_{33} \wedge q_{34})$
$= \max (0.4, 0.6, 0.5)$
$= 0.6$
$\therefore R = [r_{ij}]$

$$R = \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}_{3 \times 4}$$

* Relational Join of Fuzzy Relations

let $P(X,Y)$ & $Q(Y,Z)$ be the two fuzzy relations. The relational join of P & Q is ternary fuzzy relational defined by

$R(X,Y,Z) = P(X,Y) * Q(Y,Z)$
$R(x,y,z) = (P*Q)(x,y,z)$
$R(x,y,z) = \min \{ P(x,y), Q(y,z) \}$

* Ex. If $X = \{1,2,3,4\}$, $Y = \{a,b,c\}$, $Z = \{\alpha, \beta\}$ & If $P(X,Y)$ & $Q(Y,Z)$ be a fuzzy relation given by,
$$P = \frac{0.7}{1,a} + \frac{0.5}{1,b} + \frac{1}{2,a} + \frac{1}{3,b} + \frac{0.4}{4,b} + \frac{0.3}{4,c}$$
$$Q = \frac{0.6}{a,\alpha} + \frac{0.8}{a,\beta} + \frac{1}{b,\beta} + \frac{0.9}{c,\beta}$$
find $P \circ Q$ & $P * Q$

→ In the matrix form

$$P = \begin{array}{cc} & \begin{array}{ccc} a & b & c \end{array} \\ & \left[\begin{array}{ccc} 0.7 & 0.5 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.4 & 0.3 \end{array}\right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \qquad q = \begin{array}{cc} & \begin{array}{cc} \alpha & \beta \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \left[\begin{array}{cc} 0.6 & 0.8 \\ 0 & 1 \\ 0 & 0.9 \end{array}\right] \end{array}$$

$$P \circ Q = \begin{bmatrix} 0.6 & 0.7 \\ 0.6 & 0.8 \\ 0 & 1 \\ 0 & 0.4 \end{bmatrix}$$

$P * Q$ is ternary relation on $X \times Y$.

$X \times Y \times Z = \{(1,a,\alpha), (1,a,\beta), (1,b,\alpha), (1,b,\beta), (1,c,\alpha), (1,c,\beta), (2,a,\alpha), (2,a,\beta), (2,b,\alpha), (2,b,\beta), (2,c,\alpha), (2,c,\beta), (3,a,\alpha)\ (3,a,\beta), (3,b,\alpha), (3,b,\beta), (3,c,\alpha), (3,c,\beta), (4,a,\alpha)\ (4,a,\beta)\ (4,b,\alpha)\ (4,b,\beta)\ (4,c,\alpha)\ (4,c,\beta)\}$

$$(P * Q)(1,a,\alpha) = \min\{P(1,a), Q(a,\alpha)\}$$
$$= \min\{0.7, 0.6\}$$
$$= 0.6.$$

$(P*Q)(1,a,\beta) = \min\{P(1,a), Q(a,\beta)\}$
$= \min\{0.7, 0.8\}$
$= 0.7$

$(P*Q)(1,b,\alpha) = \min\{P(1,b), Q(b,\alpha)\}$
$= \min\{0.5, 0\}$
$= 0$

$(P*Q)(1,b,\beta) = \min\{P(1,b), Q(b,\beta)\}$
$= \min\{0.5, 1\}$
$= 0.5$

$(P*Q)(1,c,\alpha) = \min\{P(1,c), Q(c,\alpha)\}$
$= \min\{0, 0\}$
$= 0$

$(P*Q)(1,c,\beta) = \min\{P(1,c), Q(c,\beta)\}$
$= \min\{0, 0.9\}$
$= 0$

$(P*Q)(2,a,\alpha) = \min\{P(2,a), Q(a,\alpha)\}$
$= \min\{1, 0.6\}$
$= 0.6$

$(P*Q)(2,a,\beta) = \min\{P(2,a), Q(a,\beta)\}$
$= \min\{1, 0.8\}$
$= 0.8$

$(P*Q)(2,b,\alpha) = \min\{P(2,b), Q(b,\alpha)\}$
$= \min\{0, 0\}$
$= 0$

$(P*Q)(2,b,\beta) = \min\{P(2,b), Q(b,\beta)\}$
$= \min\{0, 1\}$
$= 0$

$(P*Q)(2,c,\alpha) = \min\{P(2,c), Q(c,\alpha)\}$
$= \min\{0, 0\}$
$= 0$

$(P*Q)(2,c,\beta) = \min\{P(2,c), Q(c,\beta)\}$
$= \min\{0, 0.9\}$
$= 0$

$(P*Q)(3,a,\alpha) = \min\{P(3,a), Q(a,\alpha)\}$
$= 0$

$(P*Q)(3,a,\beta) = \min\{P(3,a), Q(a,\beta)\}$
$= 0$

$(P*Q)(3,b,\alpha) = \min\{P(3,b), Q(b,\alpha)\}$
$= 0$

$(P*Q)(3,b,\beta) = \min\{P(3,b), Q(b,\beta)\}$
$= 1$

$(P*Q)(3,c,\alpha) = \min\{P(3,c), Q(c,\alpha)\}$
$= 0$

$(P*Q)(3,c,\beta) = \min\{P(3,c), Q(c,\beta)\}$
$= 0$

$(P*Q)(4,a,\alpha) = \min\{P(4,a), P(a,\alpha)\}$
$= 0$

$(P*Q)(4,a,\beta) = \min\{P(4,a), P(a,\beta)\}$
$= 0$

$(P*Q)(4,b,\alpha) = \min\{P(4,b), P(b,\alpha)\}$
$= 0$

$(P*Q)(4,b,\beta) = \min\{P(4,b), P(b,\beta)\}$
$= 0.4$

$(P*Q)(4,c,\alpha) = \min\{P(4,c), P(c,\alpha)\}$
$= 0$

$(P*Q)(4,c,\beta) = \min\{P(4,c), P(c,\beta)\}$
$= 0.3$

→ • Remark

note that,

$(P*Q)(x,y,z) = \min\{P(x,y), Q(y,z)\}$

$(P \circ Q)(x,z) = \max_{y \in Y} \min\{P(x,y), Q(y,z)\}$

$(P \circ Q)(x,z) = \max_{y \in Y} (P*Q)(x,y,z)$

* Binary fuzzy relation on X

A binary fuzzy relation on X is a fuzzy subset of $X \times X$. If R is binary fuzzy relation on X then $R: X \times X \to I$.

for any $x,y \in X$

$R(x,y)$ indicates the degree of relationship between x & y. [~~$R(x,y)=1$, x is related to y~~]

[$R(x,y)=1$, x is related to y
$=0$, x is not related to y.
$R(x,x)=1 \quad \forall x \in X$]

• Def^n:-

let R be a binary fuzzy relation on X

1) R is reflexive iff $R(x,x)=1, \forall x \in X$.

2) R is irreflexive, iff $R(x,x) \neq 1 \quad \forall x \in X$

3) R is antireflexive, iff $R(x,x) \neq 1$
for some $x \in X$

4) R is ϵ-reflexive iff $R(x,x) \geq \epsilon \quad \forall x \in X$
where, $0 \leq \epsilon \leq 1$

Note that, for $\epsilon = 1$, ϵ-reflexivity $\Rightarrow$ reflexivity

• Def^n:-

let R be a binary fuzzy relation on X then

① R is symmetric iff $R(x,y) = R(y,x) \quad \forall x,y \in X$

② R is asymmetric iff $R(x,y) \neq R(y,x)$ ~~$\forall x,y \in$~~
for some $x,y \in X$

③ R is antisymmetric is $R(x,y) > 0$ & $R(y,x) > 0$
$\Rightarrow x = y$

• Def^n:-

If R is a binary fuzzy relation on X×Y,
then we defined ~~magnitude of R~~ by,

$$|R| = \sum_{x,y \in X} R(x,y)$$

& $R^{-1}(x,y) = R(y,x)$

* Eg.

1) let R be a fuzzy relation on $X = \{x_1, x_2, x_3\}$

$$R = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.3 & 1 & 0 \\ 0.7 & 0 & 1 \end{bmatrix}$$

then $R(x,x) = 1 \quad \forall x \in X$

i.e, R is reflexive.

since the matrix for R is symmetric so the fuzzy relation R is symmetric

2) $R = \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$

$R(x,x) \neq 1$, for some $x \in X$

i.e. R is antireflexive.

R is ϵ - reflexive, ~~for~~ $\forall\ \epsilon \leq 0.2$

& R is antisymmetric

3) $R = \begin{bmatrix} 1 & 0 & 0.8 & 0 \\ 0 & 1 & 0 & 0.6 \\ 0.8 & 0 & 1 & 0.8 \\ 0 & 0.8 & 0.8 & 1 \end{bmatrix}$

R is reflexive

R is ϵ-reflexive, for $\epsilon = 1$

R is antisymmetric

- Def^n

A fuzzy relation R on X is called transitive if,

$$R(x,z) \geq \max_{y \in X} \min \{R(x,y), R(y,z)\} \quad \forall (x,z) \in X \times Y$$

this transitivity is also called maxmin transitivity

If the above inequality fails for some elements in $X \times X$ then the fuzzy relation R is called not-transitive.

If $R(x,z) < \max_{y \in Y} \min \{R(x,y), R(y,z)\}$

$\forall (x,z) \in X \times X$

then the fuzzy relation R is antitransitive

• Def^n :-

1) A fuzzy relation on X which is reflexive, symmetric & transitive is called an equivalence relation or similarity relation.

2) Reflexive & symmetric fuzzy relation on R is called compatibility or tolerence relation

3) Reflexive, antisymmetric & transitive fuzzy relation is called partial order fuzzy relation.

4) Symmetric & transitive fuzzy relation is called quasi equivalence relation.

5) reflexive & transitive fuzzy relation is called preordering or quasi ordering fuzzy relation

6) Antireflexive, Antisymmetric & transitive fuzzy relation is called strict ordering fuzzy relation.

e.g.

1) let R be fuzzy relation on $X = \{x_1, x_2, x_3\}$ given by,

$$R = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

R is reflexive. since $R(x,x) = 1$

R is symmetric $\because R(x,y) = R(y,x)$.

$R(x_1, x_2) = 0.5$

$\max \min \{ R(x, y),\ R(y, x) \}$

$= \max \{ R(x_1, x_1) \wedge R(x_1, x_2),\ R(x_1, x_2) \wedge R(x_2, x_2),\ R(x_1, x_3) \wedge R(x_3, x_2) \}$

$= \max \{ 0.5, 0.5, 0.7 \}$

$= 0.7$

$\therefore R(x_1, x_2) \not\geq \max\limits_{y} \min \{ R(x, y), R(y, x_2) \}$

Hence R is not transitive.

2) let R be fuzzy relation on $X \times Y$

$$R = \begin{array}{c|c} & \begin{matrix} y_1 & y_2 & y_3 & y_4 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.3 & 0.6 & 0 & 1 \\ 0.7 & 0 & 1 & 0.5 \\ 0.5 & 0 & 0 & 0.2 \end{bmatrix} \end{array}$$

find ran R, dom R, |R|, h(R) & R^{-1}

$\rightarrow$

$$R = \begin{bmatrix} 0.3 & 0.6 & 0 & 1 \\ 0.7 & 0 & 1 & 0.5 \\ 0.5 & 0 & 0 & 0.2 \end{bmatrix}$$

~~The sagittal diagram of R given by.~~

$\text{Ran } R(y) = \max\limits_{x} R(x, y)$

$\text{ran } R(y_1) = \max\limits_{x} \{ R(x_1, y_1),\ R(x_2, y_1),\ R(x_3, y_1) \}$

$= 0.7$

$\text{ran } R(y_2) = \max\limits_{x} \{ 0.6, 0, 0 \}$

$= 0.6.$

$\text{ran } R(y_3) = \max_x R(x, y_3)$

$= \max\{0, 1, 0\}$

$= 1$

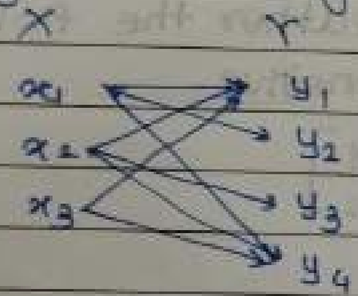
$\text{ran } R(y_4) = \max_x R(x, y_4)$

$= \max\{1, 0.5, 0.2\}$

$= 1$

$$\text{ran}(R) = \frac{0.7}{y_1} + \frac{0.6}{y_2} + \frac{1}{y_3} + \frac{1}{y_4}$$

The sagittal diagram of R given by



$\text{dom } R(x) = \max_y R(x, y)$

$\text{dom } R(x_1) = \max_y R(x_1, y)$

$= \max\{0.3, 0.6, 0, 1\}$

$= 1$

$\text{dom } R(x_2) = \max R(x_2, y)$

$= \max\{0.7, 0, 1, 0.5\}$

$= 1$

$\text{dom } R(x_3) = \max R(x_3, y)$

$= \max\{0.5, 0, 0, 0.2\}$

$= 0.5$

$$\text{dom } R = \frac{1}{x_1} + \frac{1}{x_2} + \frac{0.5}{x_3}$$

$$h(R) = \max_{x,y} R(x,y)$$

$$= 1$$

$$R^{-1} = \begin{bmatrix} 0.3 & 0.7 & 0.5 \\ 0.6 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0.5 & 0.2 \end{bmatrix}$$

$|R| = 0.3 + 0.6 + 0 + 1 + 0.7 + 0 + 1 + 0.5 + 0.5 +$

$|R| = 4.8$

Q. Verify that weather the following fuzzy relation are transitive.

1) $R = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 0.2 \end{bmatrix}$

2) $R = \begin{bmatrix} 1 & 0 & 0.6 \\ 0.5 & 1 & 0.5 \\ 0.6 & 0.5 & 1 \end{bmatrix}$

1) $R = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 0.2 \end{bmatrix}$

R is irreflexive iff $R(x,x) \neq 1, \forall x \in X$

R is

* Def^n:-
let R be binary fuzzy relation on X. A transitive closure of R is a fuzzy relation on X containing R & which is maximum minimum transitive.
we denote max min transitive closure of R by R_T
The following algorithm given by step by step process of finding transitive closure of R

Step I :-
$R' = R \cup (R \circ R)$

Step II :-
If $R' \neq R$ then $R' = R$ & go to step I

Step III :-
$R' = R_T$

* Example
1) let R be fuzzy relation given by

$$R = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

clearly R is not transitive
∴ we obtain its transitive closure

Step I
consider

$$R \circ R = \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$R' = R \cup (R \circ R)$

$$= \begin{bmatrix} 1 & 0.5 & 0.8 \\ 0.5 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$R' \neq R.$

Step II :

let $R' = R = \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$

$$R \circ R = \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$R' = R \cup (R \circ R)$

$$= \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

$R' = R_T$

step: III $R^1 \neq R$

$R^1 = R_T$

$$R_T = \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$$

Also $R \subseteq R_T$ & R_T is transitive.

* 2) weather the following fuzzy relation are transitive if not find the transitive closure.

① $\begin{bmatrix} 0.6 & 0.6 & 0 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0 \end{bmatrix}$

→ let

$$\begin{array}{cc} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ R = \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.6 & 0.6 & 0 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0 \end{bmatrix} \end{array}$$

$R(x_1, x_2) = 0.6$

$\max_y \min \{ R(x_1, y), R(y, x_2) \}$

$= \max_y \{ R(x_1, x_1) \wedge R(x_1, x_2), R(x_1, x_2) \wedge R(x_2, x_2), R(x_1, x_3) \wedge R(x_3, x_2) \}$

$= \max_y \{ 0.6, 0.6, 0 \}$

$= 0.6$

∴ R is transitive

$R(x_1, x_3) = 0$

$\max_y \min \{ R(x_1, y), R(y, x_3) \}$

$$= \max_y \{ R(x_1,x_1) \wedge R(x_1,x_3),\ R(x_1,x_2) \wedge R(x_2,x_3),\ R(x_1,x_3) \wedge R(x_3,x_3) \}$$

$$= \max_y \{0, 0.1, 0\}$$

$$= 0.1$$

$$\neq R(x_1, x_3)$$

$\therefore$ R is not transitive

$\therefore$ we find its transitive closure.

Step I : consider

$$R \circ R = \begin{bmatrix} 0.6 & 0.6 & 0 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0 \end{bmatrix} \begin{bmatrix} 0.6 & 0.6 & 0 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

let $R' = R \cup (R \circ R)$

$$= \begin{bmatrix} 0.6 & 0.6 & 0 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

$\therefore R' \neq R$

Step II :

let $R' = R = \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$

$$R \circ R = \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

$R^1 = R \cup (R \circ R)$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

$R^1 = R$

step III : $R^1 = R^T$

$$R^T = \begin{bmatrix} 0.6 & 0.6 & 0.1 \\ 0 & 0.6 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix}$$

Also,

$R \subseteq R_T$ & R_T is transitive.

2)
$$\begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$$

<u>Solⁿ :-</u>

let $R = \begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$

$R(x_1, x_3) = 0$

$\max_y \min \{ R(x_1, y), R(y, x_3) \}$

$= \max_y \{ R(x_1, x_1) \wedge R(x_1, x_3), R(x_1, x_2) \wedge R(x_2, x_3), R(x_1, x_3) \wedge R(x_3, x_3), R(x_1, x_4) \wedge R(x_4, x_3) \}$

$= \max \{0, 0, 0, 0\}$

$= 0$

$\therefore$ R is transitive.

$R(x_2, x_4) = 1$

$\max_y \min \{ R(x_2, y), R(y, x_4) \}$

$= \max_y \{ R(x_2, x_1) \wedge R(x_1, x_4), R(x_2, x_2) \wedge R(x_2, x_4), R(x_2, x_3) \wedge R(x_3, x_4), R(x_2, x_4) \wedge R(x_4, x_4) \}$

$= \max \{0, 0, 0, 0\}$

$= 0$

$\neq R(x_2, x_4)$

$\therefore$ R is not transitive

$\therefore$ we find it's transitive closure.

Step I :

Consider,

$$ROR = \begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.4 \\ 0 & 0.4 & 0 & 0 \end{bmatrix}$$

let $R' = R \cup (ROR)$

$$= \begin{bmatrix} 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.8 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0 & 0.8 & 0 \\ 0 & 0 & 0 & 0.4 \\ 0 & 0.4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.8 & 0 \end{bmatrix}$$

$\therefore R' \neq R$

Step II :

let $R' = R = \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.8 & 0 \end{bmatrix}$

$$ROR = \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.8 & 0 \end{bmatrix} \circ \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.8 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.5 & 0.7 & 0.5 \\ 0 & 0.4 & 0.8 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \end{bmatrix}$$

$R' = R \cup (ROR)$

$$= \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0 & 0.8 & 1 \\ 0 & 0.4 & 0 & 0.4 \\ 0 & 0.4 & 0.8 & 0 \end{bmatrix} \vee \begin{bmatrix} 0.7 & 0.5 & 0.7 & 0.5 \\ 0 & 0.4 & 0.8 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7 & 0.5 & 0.8 & 0.5 \\ 0 & 0.4 & 0.8 & 1 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.8 & 0.4 \end{bmatrix}$$

$\therefore R' \neq R$

$$R' = R = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 1 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.8 & 0.4 \end{bmatrix}$$

$$ROR = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.4 & 0.4 \end{bmatrix}$$

$$R' = R \cup (R \circ R) = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 1 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.8 & 0.4 \end{bmatrix}$$

$R = R'$

Step III : $R' = R_T$

$$\therefore R^T = \begin{bmatrix} 0.7 & 0.5 & 0.5 & 0.5 \\ 0 & 0.4 & 0.8 & 1 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0.4 & 0.8 & 0.4 \end{bmatrix}$$

Also, $R \subseteq R_T$, & R_T is transitive.

3]

let $R = \begin{bmatrix} 0 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0 & 0.8 & 0 \\ 1 & 0 & 0.4 & 0 \\ 0 & 0.3 & 0 & 1 \end{bmatrix}$

$R(x_1, x_2) = 0.4$

$\max\limits_{y} \min \{ R(x_1, y), R(y, x_2) \}$

$= \max \{$

$= \max \{0, 0, 0, 0.3\}$

$= 0.3$

$R(x_2, x_4) = 0$

$\max \{ R(x_2, x_1) \wedge R(x_1, x_4), R(x_2, x_2) \wedge R(\quad),$
$R(x_1, x_3) \wedge R(x_3, x_4), R(x_2, x_4) \wedge R(\quad) \}$

$= \max \{0.3, 0, 0, 0\}$

$= 0.3$

$\therefore$ R is not transitive.

Now, we find transitive closure.

Step I: $R^1 = R \cup (R \circ R)$

$$R \circ R = \begin{bmatrix} 0.3 & 0.3 & 0.4 & 0.3 \\ 0.8 & 0.3 & 0.4 & 0.3 \\ 0.4 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

$$R^1 = R \cup (R \circ R) = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.3 \\ 0.8 & 0.3 & 0.8 & 0.3 \\ 1 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

$R \neq R^1$

Step II:

$$R'' = R = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.3 \\ 0.8 & 0.3 & 0.8 & 0.3 \\ 1 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

$$R \circ R = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.3 \\ 0.8 & 0.4 & 0.4 & 0.3 \\ 0.4 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

$$R = R' \cup R \circ R = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.3 \\ 0.8 & 0.4 & 0.8 & 0.3 \\ 1 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

$\therefore R = R^1$

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step III : $R' = R_T$

$$R_T = \begin{bmatrix} 0.4 & 0.4 & 0.4 & 0.3 \\ 0.8 & 0.4 & 0.4 & 0.3 \\ 1 & 0.4 & 0.4 & 0.3 \\ 0.3 & 0.3 & 0.3 & 1 \end{bmatrix}$$

Also, $R \subseteq R_T$ & R_T is transitive.

* Fuzzy Equivalence Relation :-

A binary fuzzy relation R on X which is reflexive, symmetric & transitive is called a fuzzy equivalence relation or similarity relation.

for each element $x \in X$, we define similarity class of x as a fuzzy set $A_x : X \longrightarrow I$

$$A_x(y) = R(x, y)$$

e.g.

Ex ① let $R = \begin{bmatrix} 1 & 0.7 & 0.8 \\ 0.7 & 1 & 0.7 \\ 0.8 & 0.7 & 1 \end{bmatrix}$

be the fuzzy relation on $X = \{x_1, x_2, x_3\}$ then

$A_{x_1} : X \longrightarrow I$

$A_{x_1}(y) = R(x, y)$

$A_{x_1}(x_1) = R(x_1, x_1) = 1$

$A_{x_1}(x_2) = R(x_1, x_2) = 0.7$

$A_{x_1}(x_3) = R(x_1, x_3) = 0.8$

$$A_{x_1} = \frac{1}{x_1} + \frac{0.7}{x_2} + \frac{0.8}{x_3}$$

$A_{x_2}(y) = R(x_2, y)$

$Ax_2(x_1) = R(x_2, x_1) = 0.7$

$Ax_2(x_2) = R(x_2, x_2) = 1$

$Ax_2(x_3) = R(x_2, x_3) = 0.7$

$\therefore Ax_2 = \frac{0.7}{x_1} + \frac{1}{x_2} + \frac{0.7}{x_3}$

$Ax_3(y) = R(x_3, y)$

$Ax_3(x_1) = R(x_3, x_1) = 0.8$

$Ax_3(x_2) = R(x_3, x_2) = 0.7$

$Ax_3(x_3) = R(x_3, x_3) = 1$

$\therefore Ax_3 = \frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{1}{x_3}$

* α-cut of fuzzy Relation :-

let R be a fuzzy relation on X then for $\alpha \in [0,1]$ we define,

$$^{\alpha}R = \{(x,y) \,/\, R(x,y) \geq \alpha\}$$

eg.

In the above example.

$^{0.8}R = \{(x_1, x_3), (x_1, x_1), (x_2, x_2), (x_3, x_1), (x_3, x_3)\}$

α-cut is also an equivalence relation.

* Proposition :-

Statement :- If R is similarity relation on X then α-cut of R is a crisp equivalence relation on X, $\forall \alpha \in I$

proof :-

let R be similarity relation on X.

i.e., $R : X \times X \to I$ is reflexive, symmetric & transitive.

for each $\alpha \in [0,1]$

The α - cut of R is given by

$\alpha_R = \{(x,y) / R(x,y) \geq \alpha\}$

we prove that α_R is an equivalence relation.

Now, for each $x \in X$

we have,

$R(x,x) = 1$

hence, $R(x,x) \geq \alpha$, $\forall \ \alpha \in [0,1]$

i.e, $(x,x) \in \alpha_R$

i.e, α_R is reflexive.

If $(x,y) \in \alpha_R$

$\Rightarrow R(x,y) \geq \alpha$

$\Rightarrow R(y,x) \geq \alpha$

$\Rightarrow (y,x) \in \alpha_R$

i.e, α_R is symmetric

Next,

if $(x,y) \in \alpha_A$ & $(y,z) \in \alpha_R$

$\Rightarrow R(x,y) \geq \alpha$ & $R(y,z) \geq \alpha$

$R(x,y) \cap R(y,z) \geq \alpha$, $\forall \ y \in X$

$\Rightarrow \max\limits_{y} (R(x,y) \cap R(y,z)) \geq \alpha$

$\Rightarrow R(x,z) \geq \max\limits_{y} (R(x,y) \cap R(y,z)) \geq \alpha$

$\Rightarrow R(x,z) \geq \alpha$

$\Rightarrow (x,z) \in \alpha_R$

i.e, α_A is transitive.

$\therefore \alpha_R$ is reflexive, symmetric & transitive

& hence, it is an equivalence relation, $\forall \ \alpha$

* Defⁿ :-

If R is similarity relⁿ on X then each α_R is a crisp equivalence relation on X.
The partition of X corresponding to equivalence relⁿ α_R is denoted by $\pi(\alpha_R)$.
The set of all partition of X corresponding to the similarity relⁿ R is denoted by $\pi(R)$.
Thus,

$\pi(R) = \{\pi(\alpha_R) / \text{ for each } \alpha \in [0,1]\}$

further, this set of partition $\pi(R)$ is a nested condition.

i.e, $\alpha \geq \beta$

$\Rightarrow \pi(\alpha_R)$ is a refinement of $\pi(\beta_R)$

Ex1) let $X = \{a, b, c, d, e, f, g\}$

let R be a similarity relⁿ on X is given by,

$$R = \begin{bmatrix} 1 & 0.8 & 0 & 0.4 & 0 & 0 & 0 \\ 0.8 & 1 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0.9 & 0.5 \\ 0.4 & 0.4 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0.9 & 0.5 \\ 0 & 0 & 0.9 & 0 & 0.9 & 1 & 0.5 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0.5 & 1 \end{bmatrix}$$

The distinct level cut of R are obtained from $\alpha = \{0.4, 0.5, 0.8, 0.9\}$

Solution :-

i) $\alpha = 0.4$

$0.4_R = \{ (a,a), (a,b), (a,d), (b,a), (b,b), (b,d), (c,c), (c,f), (c,g), (c,e), (d,a), (d,b), (d,d), (e,c), (e,e), (e,f), (e,g), (f,c), (f,e) (f,f), (f,g), (g,c), (g,e), (g,f), (g,g) \}$ rel^n on X.

0.4_R is a equivalence rel^n on X.
The equivalence classes corresponding to this equivalence relation are $\{a,b,d\}$, $\{c,e,f,g\}$
Hence, partition of X corresponding to 0.4_R is
$\therefore \pi(0.4) = \{ \{a,b,d\}, \{c,e,f,g\} \}$.

ii) $0.5 = x$
$0.5_R = \{ (a,a), (a,b), (b,a) (b,b), (c,c), (c,e), (c,f), (c,g), (d,d), (e,c), (e,e), (e,f), (e,g), (f,c), (f,e) (f,f), (f,g), (g,c), (g,e), (g,f), (g,g) \}$

0.5_R is an equivalence rel^n on X.
Hence, partition of X corresponding to 0.5_R is $\{ \{a,b\}, \{c,e,f,g\} \}$
i.e. $\pi(0.5) = \{ \{a,b\}, \{c,e,f,g\}, \{d\} \}$

iii) $x = 0.8$
$0.8_R = \{ (a,a), (a,b), (b,a), (b,b) (c,c), (c,e) (c,f), (d,d), (e,c), (e,e), (e,f), (f,c), (f,e), (f,f), (g,g) \}$

$\therefore 0.8_R$ is an equivalence rel^n on X
Hence, partition of X corresponding to 0.8_R is $\{ \{a,b\}, \{e,f\}, \{d\}, \{g\} \}$
$\therefore \pi(0.8) = \{ \{a,b\}, \{e,f\}, \{d\}, \{g\} \}$

iv) $\alpha = 0.9$

$0.9_R = \{ (a,a), (b,b), (c,c), (c,e), (c,f), (d,d), (e,c), (e,e), (e,f), (f,c), (f,e), (f,f), (g,g) \}$

0.9_R is equivalence relⁿ on X.

Hence, partition of X corresponding to 0.9_R is $\{ \{a\}, \{b\}, \{c,e,f\}, \{d\}, \{g\} \}$

$\therefore \pi(0.9) = \{ \{a\}, \{b\}, \{c,e,f\}, \{d\}, \{g\} \}$

v) $\alpha = 1$

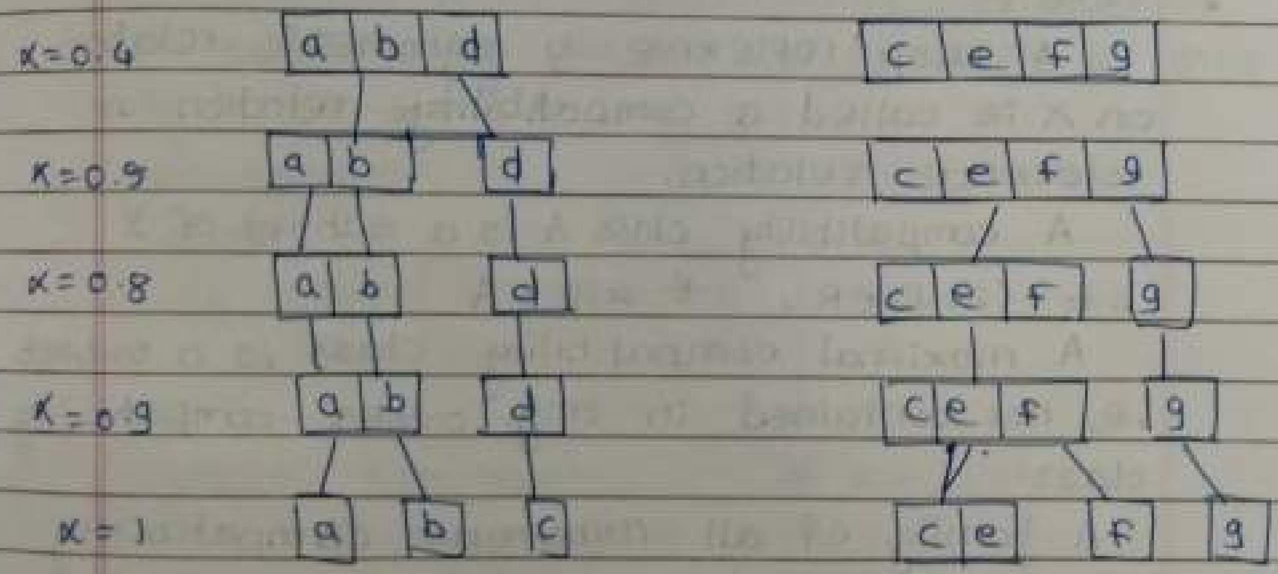
$1_R = \{ (a,a), (b,b), (c,c), (c,e), (d,d), (e,c), (e,e), (f,f), (g,g) \}$

$\therefore 1_R$ is an equivalence relⁿ on X.

Hence, partition of X corresponding to 1_R is $\{ \{a\}, \{b\}, \{c,e\}, \{d\}, \{f\}, \{g\} \}$

$\therefore \pi(1) = \{ \{a\}, \{b\}, \{c,e\}, \{d\}, \{f\}, \{g\} \}$

The partition tree for this relⁿ R on X is given by



* Note :-

1) The equivalence classes formed by different levels of refinement of a similarity relation can be interpretete as a grouping the elements that are similar to each other w.r.t level α.

MCQ 2) Fuzzy equivalence relation is a cut worthy property that is it preserves property.

MCQ 3) fuzzy reflexivity, symmetric & max min transitivity are also cut worthy property

* * Fuzzy compatibility relation :-

A fuzzy reflexive & symmetric relation R (X,X) is called fuzzy compatibility or proximity relation.

* • Note :-

A crisp reflexive & symmetric relation on X is called a compatibility relation or tolerence relation.

A compatibility class A is a subset of X s.t $(x,y) \in R$, $\forall\ x,y \in A$

A maximal compatibility class is a subset i.e, not contained in any other compatibility class.

A family of all maximum compatibility classes is called a complete cover of X w.r.t. R.

* Def^n:-

let R be fuzzy compatibility relation on X & α-compatibility class is a subset A of X s.t. $R(x, y) \geq \alpha$, $\forall\, x, y \in A$

* Def^n:-

The maximal α-compatibility class is a subset of X which is not contained in any other class.

* Def^n:-

The collection of all maximal α-compatibility class is called a complete cover of X.

* Ex 1] let $X = \{a, b, c\}$ & R be fuzzy rel^n on X given by $R = \begin{bmatrix} 1 & 0.4 & 0 \\ 0.4 & 1 & 0.7 \\ 0 & 0.7 & 1 \end{bmatrix}$

show that R is compatibility rel^n. Find α-compatibility classes & complete α-cover of X for $\alpha = 0.4, 0.7$ & 1

Solution,

Given,

$$R = \begin{bmatrix} 1 & 0.4 & 0 \\ 0.4 & 1 & 0.7 \\ 0 & 0.7 & 1 \end{bmatrix}$$

Since, $R(x, x) = 1$, $\forall\, x \in X$

$\therefore$ R is reflexive

By symmetry of the matrix

R is symmetric fuzzy rel^n.

Now,

$R(a,c) = 0$

$\max_y \min \{ R(a,y), R(y,c) \}$

$= \max \{ R(a,a) \wedge R(a,c),\ R(a,b) \wedge R(b,c)$
$R(a,c) \wedge R(c,c) \}$

$= \max \{ 0, 0.4, 0 \}$

$= 0.4$

$R(a,c) \ngeq \max_y \min \{ R(a,y), R(y,c) \}$

$\therefore$ R is not transitive.

$\therefore$ R is compatibility reln (proximity reln)

Given, $X = \{a,b,c\}$

$$R = \begin{bmatrix} 1 & 0.4 & 0 \\ 0.4 & 1 & 0.7 \\ 0 & 0.7 & 1 \end{bmatrix}.$$

for $\alpha = 0.4$

α - compatibility classes are

$A_1 = \{a\}$, $A_2 = \{a,b\}$, $A_3 = \{b\}$, $A_4 = \{b\ c\}$, $A_5 = \{c$

A_2 & A_4 are maximal α - compatibility class

for $\alpha = 0.4$

The complete α - cover of X for $\alpha = 0.4$ is

$\{ \{a,b\}, \{b,c\} \}$

for $\alpha = 0.7$

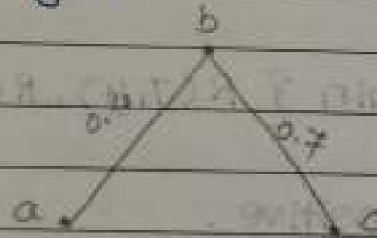
α - compatibility classes are

$A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{b,c\}$, $A_4 = \{c\}$

A_1 & A_3 is maximal α- compatibility class for $\alpha = 0.7$
The complete α-cover of X for $\alpha = 0.7$ is $\{\{a\}, \{b,c\}\}$

for $\alpha = 1$
α-compatibility classes are
$A_1 = \{a\}$, $A_2 = \{b\}$, $A_3 = \{c\}$
A_1, A_2, A_3 are maximal α- compatibility class for $\alpha = 1$
The complete α-cover of X for $\alpha = 1$ is
$\{\{a\}, \{b\}, \{c\}\}$

- Note :-
Compatibility relation can be represented by an undirected graph in the above ex. the fuzzy reln can be represented by the following dia.



2] let $X = \{1, 2, \ldots, 9\}$

$$R = \begin{bmatrix} 1 & 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0.8 & 0.7 & 0.5 & 0 & 0 \\ 0 & 0 & 0.8 & 0.8 & 1 & 0.7 & 0.5 & 0.7 & 0 \\ 0 & 0 & 0 & 0.7 & 0.7 & 1 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0.4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

i) show that R is reflexive & symmetric (non-transitive)

ii) Find α-compatibility classes for $\alpha = 0.4$, 0.5, 0.7, 0.8, 1

iii) Find maximal α-compatibility classes for each α & hence obtain complete α-cover of X

iv) Also draw a graph for this relation.

Solution :-

$R(x,x) = 1$, $\forall x \in X$

$\therefore$ R is reflexive

By symmetry of matrix,

R is symmetric fuzzy relation,

$R(7,3) = 0$

$\max_y \min \{R(7,y), R(y,3)\}$

$= \max \{0, 0, 0, 0.5, 0.5, 0.4, 0\}$

$= 0.5$

$R(7,3) \neq \max_y \min \{R(7,y), R(y,3)\}$

$\therefore$ R is not transitive.

$\therefore$ R is compatibility reln.

Given, $X = \{1, 2, \ldots, 9\}$

for $\alpha = 0.4$

α-compatibility classes are.

$A_1 = \{1\}$, $A_2 = \{1,2\}$, $A_3 = \{2\}$, $A_4 = \{3\}$,

$A_5 = \{3,4\}$, $A_6 = \{3,5\}$, $A_7 = \{4\}$, $A_8 = \{4,5\}$
$A_9 = \{4,6\}$, $A_{10} = \{4,7\}$, $A_{11} = \{5\}$, $A_{12} = \{5,6\}$
$A_{13} = \{5,7\}$, $A_{14} = \{5,8\}$, $A_{15} = \{6\}$, $A_{16} = \{6,7\}$
$A_{17} = \{7\}$, $A_{18} = \{8\}$, $A_{19} = \{9\}$

$A_2, A_5, A_6, A_8, A_9, A_{10}, A_{12}, A_{13}, A_{14}, A_{16}, A_{19}$ are maximal α - compatibility class for $\alpha = 0.4$

The complete α - cover of X for $\alpha = 0.4$ is $\{\{1,2\}, \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\}, \{5,8\}, \{6,7\}, \{9\}\}$

$\alpha = 0.5$

α - compatibility classes are

$A_1 = \{1\}$, $A_2 = \{1,2\}$, $A_3 = \{2\}$, $A_4 = \{3\}$,
$A_5 = \{3,4\}$, $A_6 = \{3,5\}$, $A_7 = \{4\}$, $A_8 = \{4,5\}$,
$A_9 = \{4,6\}$, $A_{10} = \{4,7\}$, $A_{11} = \{5\}$, $A_{12} = \{5,6\}$,
$A_{13} = \{5,7\}$, $A_{14} = \{5,8\}$, $A_{15} = \{6\}$, $A_{16} = \{7\}$,
$A_{17} = \{8\}$, $A_{18} = \{9\}$

$A_2, A_5, A_6, A_8, A_9, A_{10}, A_{12}, A_{13}, A_{14}, A_{18}$ are maximal α - compatibility class for $\alpha = 0.5$

The complete α - cover of X for $\alpha = 0.5$ is $\{\{1,2\}, \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\}, \{5,8\}, \{9\}\}$

for $\alpha = 0.7$

α - compatibility classes are

$A_1 = \{1\}$, $A_2 = \{1,2\}$, $A_3 = \{2\}$, $A_4 = \{3\}$,
$A_5 = \{3,4\}$, $A_6 = \{3,5\}$, $A_7 = \{4\}$, $A_8 = \{4,5\}$
$A_9 = \{4,6\}$, $A_{10} = \{5\}$, $A_{11} = \{5,6\}$, $A_{12} = \{5,8\}$

$A_{13} = \{6\}$, $A_{14} = \{7\}$, $A_{15} = \{8\}$, $A_{16} = \{9\}$

$A_2, A_5, A_6, A_8, A_9, A_{11}, A_{12}, A_{15}$ are maximal α-compatibility classes for $\alpha = 0.7$

The complete cover of X for $\alpha = 0.7$ is

$\{ \{1,2\}, \{3,4\}, \{3,5\}, \{4,5\}, \{4,6\}, \{5,6\}, \{5,8\}, \{9\} \}$

for $\alpha = 0.8$

α-compatibility classes are

$A_1 = \{1\}$, $A_2 = \{1,2\}$, $A_3 = \{2\}$, $A_4 = \{3\}$,
$A_5 = \{3,4\}$, $A_6 = \{3,5\}$, $A_7 = \{4\}$, $A_8 = \{4,5\}$,
$A_9 = \{5\}$, $A_{10} = \{6\}$, $A_{11} = \{7\}$, $A_{12} = \{8\}$, $A_{13} = \{9\}$

$A_2, A_5, A_6, A_8, A_{13}$ are maximal α-compatibility classes for $\alpha = 0.8$

The complete cover of X for $\alpha = 0.8$ is

$\{ \{1,2\}, \{3,4\}, \{3,5\}, \{4,5\}, \{9\} \}$

for $\alpha = 1$

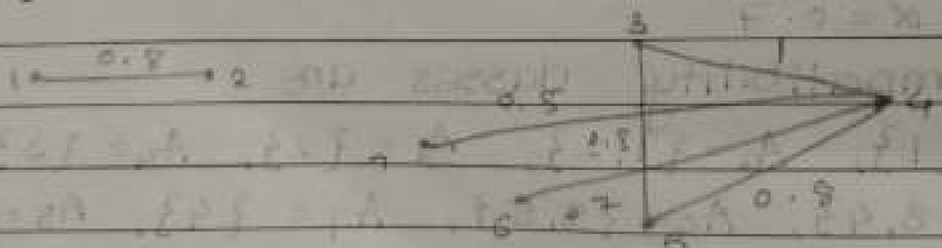
α-compatibility classes are

$A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, $A_4 = \{3,4\}$, $A_5 = \{4$
$A_6 = \{5\}$, $A_7 = \{6\}$, $A_8 = \{7\}$, $A_9 = \{8\}$, $A_{10} = \{9$

$A_1, A_2, A_4, A_6, A_7, A_8, A_9, A_{10}$ are maximal α-compatibility classes for $\alpha = 1$

The complete cover of X for $\alpha = 1$ is

$\{ \{1\}, \{2\}, \{3,4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\} \}$



* Fuzzy partial ordering relation :-

Defⁿ :-

A fuzzy binary relation R on a set X is called fuzzy partial ordering if it is reflexive, antisymmetric & transitive (max min transitive)

- Defⁿ :-

MCQ let R be partial ordering on X for every element $x \in X$, we define a dominating class of X by,

$$R_{\geq}[x](y) = R(x,y) \quad , \forall y \in X$$

i.e. $R_{\geq}[x] : X \longrightarrow I$

similarly, we define a class dominated by x as

$$R_{\leq}[x](y) = R(y,x)$$

i.e. $R_{\leq}[x] : X \longrightarrow I$

- Defⁿ :-

An element $x \in X$ is called undominated element if $R(x,y) = 0$, $\forall y \in X \quad (x \neq y)$

The element $x \in X$ is called undominating if $R(y,x) = 0$, $\forall y \in X \quad (x \neq y)$

* Defⁿ :-

let A be a crisp subset of X. let R be a fuzzy partial ordering relation on X

The fuzzy upper bond of A is a

fuzzy set denoted by, $U(R,A)$ &
it is given by,

$$U(R,A) = \bigcap_{x \in A} R_{\geq}[x]$$

where, the intersection is a t-norm on X.

A fuzzy least upper bound of set A is a unique element in $U(R,A)$ s.t $U(R,A)(x) > 0$ & $R(x,y) > 0$ $\forall y \in$ support $[U(R,A)]$

* Fuzzymorphism :-

1) let R be a crisp relation on X & Q be a crisp relation on Y. A function $h: X \to Y$ is called a homomorphism from (X,R) to (Y,Q) if $(x,y) \in R \Rightarrow (h(x), h(y)) \in Q$.

2) let $R(X,X)$ & $Q(Y,Y)$ be the fuzzy relation & let $h: X \longrightarrow Y$ be a function then h is called homomorphism of (X,R) to (Y,Q) if $R(x_1, x_2) \leq Q(h(x_1), h(x_2))$

Ex.1] let $X = \{a,b,c,d\}$ & $Y = \{\alpha, \beta, \gamma\}$, let R & Q be the fuzzy relation on X & Y resp.

$$R = \begin{bmatrix} 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0.9 & 0 \\ 1 & 0 & 0 & 0.9 \\ 0 & 0.6 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.5 & 0.9 & 0 \\ 1 & 0 & 0.9 \\ 0.6 & 0.9 & 0 \end{bmatrix}$$

let $h: X \to Y$ be a function defined by $h(a) = \alpha$, $h(b) = \alpha$, $h(c) = \beta$, $h(d) = \gamma$. verify that h is a homomorphism from (X, R) to (Y, Q).

<u>Solution:</u>

i) for $(a, b) \in X \times X$

$R(a, b) = 0.5$

$Q(h(a), h(b)) = Q(\alpha, \alpha) = 0.5$

$R(a, b) \leq Q(h(a), h(b))$

ii) for $(b, c) \in X \times X$

$R(b, c) = 0.9$

$Q(h(b), h(c)) = Q(\alpha, \beta) = 0.9$

$R(b, c) \leq Q(h(b), h(c))$

iii) for $(c, a) \in X \times X$

$R(c, a) = 1$

$Q(h(c), h(a)) = Q(\beta, \alpha) = 1$

$R(c, a) \leq Q(h(c), h(a))$

iv) for $(c, d) \in X \times X$

$R(c, d) = 0.9$

$Q(h(c), h(d)) = Q(\beta, \gamma) = 0.9$

$R(c, d) \leq Q(h(c), h(d))$

v) for $(d, b) \in X \times X$

$R(d, b) = 0.6$

$Q(h(d), h(b)) = Q(\gamma, \alpha) = 0.6$

$R(d, b) \leq Q(h(d), h(b))$

Thus, h is homomorphism from (X, R) to (Y, Q).

* Supremum - i composition of fuzzy relation:

Def^n :-

Let i be a t-norm. let $P(X,Y)$ & $Q(Y,Z)$ be two fuzzy relation then sup-i composition of P & Q is a fuzzy relation on $X \times Z$ defined by.

$$(P \overset{i}{o} Q)(x,z) = \sup_{y \in Y} i\left(P(x,y), Q(y,z)\right), \quad \forall (x,z) \in X \times Z$$

when i = minimum, the above composition becomes a standard composition (max min composition).

** Proposition :-

Statement :-

let $P(X,Y)$, $Q(Y,Z)$, $R(Z,V)$, $P_j(X,Y)$ & $Q_j(Y,Z)$ $j \in J$ be the fuzzy relations.

where, J is an index set, then the following properties hold.

1) $(P \overset{i}{o} Q) \overset{i}{o} R = P \overset{i}{o} (Q \overset{i}{o} R)$

2) $P \overset{i}{o} \left(\bigcup_{j \in J} Q_j\right) = \bigcup_{j \in J} (P \overset{i}{o} Q_j)$

3) $P \overset{i}{o} \left(\bigcap_{j \in J} Q_j\right) \subseteq \bigcap_{j \in J} (P \overset{i}{o} Q_j)$

4) $\left(\bigcup_{j \in J} P_j\right) \overset{i}{o} Q = \bigcup_{j \in J} (P_j \overset{i}{o} Q)$

5) $\left(\bigcap_{j \in J} P_j\right) \overset{i}{o} Q \subseteq \bigcap_{j \in J} (P_j \overset{i}{o} Q)$

6) $(P \overset{i}{o} Q)^{-1} = Q^{-1} \overset{i}{o} P^{-1}$

proof :-

1) For any $(x, v) \in X \times V$

consider,

$$[(P \overset{i}{\circ} Q) \overset{i}{\circ} R](x, v) = \sup_{z \in Z} i[(P \overset{i}{\circ} Q)(x, z), R(z, v)]$$

$$= \sup_{z \in Z} i\left(\sup_{y \in Y} i(P(x, y), Q(y, z)); R(z, v)\right]$$

$$= \sup_{z \in Z} \sup_{y \in Y} i\left(i(P(x, y), Q(y, z)), R(z, v)\right)$$

$$= \sup_{z \in Z} \sup_{y \in Y} i\left(P(x, y), i(Q(y, z), R(z, v)\right)$$

$$= \sup_{y \in Y} i\left(P(x, y), \sup_{z \in Z} i\left(Q(y, z), R(z, v)\right)\right)$$

$$= \sup_{y \in Y} i\left(P(x, y), (Q \overset{i}{\circ} R)(y, v)\right)$$

$$= (P \overset{i}{\circ} (Q \overset{i}{\circ} R))(x, v)$$

i.e, sup-i composition is asociative.

2) for any $(x, z) \in X \times Z$

$$(P \overset{i}{\circ} (\bigcup_{j \in J} Q_j))(x, z) = \sup_{y \in Y} i\left(P(x, y), \bigcup_{j \in J} Q_j(y, z)\right)$$

$$= \sup_{y \in Y} i\left(P(x, y), \bigvee_{j \in J} Q_j(y, z)\right)$$

$$= \sup_{y \in Y} i \bigvee_{j} \left(P(x, y), Q_j(y, z)\right)$$

$$= \bigvee_{j \in J} \sup_{y \in Y} i\ (P(x, y), Q_j(y, z))$$

$$= \bigvee_{j \in J} (P \overset{i}{\circ} Q_j)(x, z)$$

$$= \bigcup_{j \in J} (P \overset{i}{\circ} Q_j)(x, z)$$

Thus, $P \overset{i}{\circ} (\bigcup_{j \in J} Q_j) = \bigcup_{j \in J} (P \overset{i}{\circ} Q_j)$

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3) For any $(x,z) \in X \times Z$

consider,

$$(P \overset{i}{\circ} (\bigcap_{j \in J} Q_j))(x,z) = \sup_{y \in Y} i(P(x,y), (\bigcap_{j \in J} Q_j)(y,z))$$

$$= \sup_{y \in Y} i(P(x,y), (\bigwedge_{j \in J} Q_j)(y,z))$$

$$= \sup_{y \in Y} \bigwedge_{j \in J} i(P(x,y), Q_j(y,z))$$

$$\leq \bigwedge_{j \in J} \sup i(P(x,y), Q_j(y,z))$$

$$\leq \bigwedge_{j \in J} (P \overset{i}{\circ} Q_j)(x,z)$$

$$\subseteq \bigcap_{j \in J} (P \overset{i}{\circ} Q_j)(x,z)$$

Thus, $P \overset{i}{\circ} (\bigcap_{j \in J} Q_j) \subseteq \bigcap_{j \in J} (P \overset{i}{\circ} Q_j)$

4) For $(x,z) \in X \times Z$

consider

$$((\bigcup_{j \in J} P_j) \overset{i}{\circ} Q)(x,z) = \sup_{y \in Y} i((\bigcup_{j \in J} P_j)(x,y), Q(y,z))$$

$$= \sup_{y \in Y} i(\bigvee_{j \in J} P_j(x,y), Q(y,z))$$

$$= \sup_{y \in Y} i \bigvee_{j \in J} (P_j(x,y), Q(y,z))$$

$$= \bigvee_{j \in J} \sup_{y \in Y} i(P_j(x,y), Q(y,z))$$

$$= \bigvee_{j \in J} (P_j \overset{i}{\circ} Q)(x,z)$$

$$= \bigcup_{j \in J} (P_j \overset{i}{\circ} Q)(x,z)$$

Thus, $(\bigcup_{j \in J} P_j) \overset{i}{\circ} Q = \bigcup_{j \in J} (P_j \overset{i}{\circ} Q)$

5) for $(x,z) \in X \times Z$

consider,

$$\left(\left(\bigcap_{j\in J} P_j\right) \dot{o}\, Q\right)(x,z) = \sup_{y\in Y} i\left(\left(\bigcap_{j\in J} P_j\right)(x,y), Q(y,z)\right)$$

$$= \sup_{y\in Y} i\left(\bigwedge_{j\in J} P_j(x,y), Q(y,z)\right)$$

$$= \sup_{y\in Y} i \bigwedge_{j\in J} \left(P_j(x,y), Q(y,z)\right)$$

$$\leq \bigwedge_{j\in J} \sup_{y\in Y} i\left(P_j(x,y), Q(y,z)\right)$$

$$\leq \bigwedge_{j\in J} (P_j \dot{o}\, Q)(x,z)$$

$$\subseteq \bigcap_{j\in J} (P_j \dot{o}\, Q)(x,z)$$

Thus, $\left(\bigcap_{j\in J} P_j\right) \dot{o}\, Q \subseteq \bigcap_{j\in J} (P_j \dot{o}\, Q)$

6) let $P(X,Y)$ & $Q(Y,Z)$ be the fuzzy relation. then $P \dot{o}\, Q$ is a fuzzy relation on $X \times Z$. Hence, $(P \dot{o}\, Q)^{-1}$ is a fuzzy relation on $Z \times X$

$\therefore$ For any $(z,x) \in Z \times X$

consider,

$$(P \dot{o}\, Q)^{-1}(z,x) = (P \dot{o}\, Q)(x,z) \quad (\because R(x,y) = R^{-1}(y,x))$$

$$= \sup_{y\in Y} i\left(P(x,y), Q(y,z)\right)$$

$$= \sup_{y\in Y} i\left(Q(y,z), P(x,y)\right)$$

$$= \sup_{y\in Y} i\left(Q^{-1}(z,y), P^{-1}(y,x)\right)$$

$$= (Q^{-1} \dot{o}\, P^{-1})(z,x)$$

Thus, $(P \dot{o}\, Q)^{-1} = Q^{-1} \dot{o}\, P^{-1}$

* * Theorem :-

Statement: sup-i composition of fuzzy relation is monotonic increasing i.e., if $P(x,y)$, $Q_i(y,z)$ $(i=1,2)$ & $R(x,v)$ be fuzzy relation & $Q_1 \subseteq Q_2$

i) $P \overset{i}{\circ} Q_1 \subseteq P \overset{i}{\circ} Q_2$

ii) $Q_1 \overset{i}{\circ} R \subseteq Q_2 \overset{i}{\circ} R$

proof :-

Since, $P(x,y)$ & $Q_i(y,z)$ $(i=1,2)$ are fuzzy rel^n.

i) $P \overset{i}{\circ} Q_1$ & $P \overset{i}{\circ} Q_2$ are fuzzy relation on $X \times Z$

$\therefore$ for any $(x,z) \in X \times Z$

consider,

$$(P \overset{i}{\circ} Q_1)(x,z) = \sup_{y \in Y} i(P(x,y), Q_1(y,z))$$

$$\leq \sup_{y \in Y} i(P(x,y), Q_2(y,z))$$

$\because Q_1 \subseteq Q_2$, $Q_1(y,z) \leq Q_2(y,z)$

$i(a,b) \leq i(a,c)$

$b \leq c$

$$\leq (P \overset{i}{\circ} Q_2)(x,z)$$

$\Rightarrow P \overset{i}{\circ} Q_1 \subseteq P \overset{i}{\circ} Q_2$

ii) $Q_1 \overset{i}{\circ} R$ & $Q_2 \overset{i}{\circ} R$ are fuzzy rel^n on $Y \times V$

$\therefore$ for any $(y,v) \in Y \times V$

consider,

$$(Q_1 \overset{i}{\circ} R)(y,v) = \sup_{z \in Z} i(Q_1(y,z), R(z,v))$$

$$\leq \sup_{z \in Z} i(Q_2(y,z), R(z,v))$$

$$\leq (Q_2 \overset{i}{\circ} R)(y,v)$$

$\Rightarrow Q_1 \overset{i}{\circ} R \subseteq Q_2 \overset{i}{\circ} R$

* Def^n :-

let $(X, \vee)$ be arbitory set. The set of fuzzy rel^n on $X^2 (X \times X)$ for a complete lattice order monoide given by,

$< F(X^2), \cup, \cap, \overset{i}{\circ} >$

where, $F(X^2) = \{ P / P : X \times X \rightarrow I \}$
= set of all binary fuzzy rel^n on X.

The '$\cup$' & then '$\cap$' represents joint $(\vee)$ & ~~com~~ meet $(\wedge)$ of the lattice & the suprimum i composition of fuzzy rel^n is the binary operation on $F(X^2)$

The above thm says that the binary operation sup-i composition is associative.

The relation E on $X \times X$ defined by,

$$E(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{if } x \neq y \end{cases}$$

is the identity of monoid w.r.t. composition.

* i-transitive fuzzy relation :-

let R be a binary fuzzy relation on X. The relation R is called i-transitive if
$R(x, z) \geq i(R(x, y), R(y, z))$

• Ex. let $X = \{x_1, x_2, x_3\}$ & R be fuzzy relation on X given by a matrix

$$R = \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$$

whetter, R is i-transitive,
where, i = product.

Sol^n :-

Here, $R(x_1, x_1) = 1$

$i(R(x_1, x_1), R(x_1, x_1)) = i(1, 1) = 1$

$i(R(x_1, x_2), R(x_2, x_1)) = i(0, 0.3) = 0$

$i(R(x_1, x_3), R(x_3, x_1)) = i(0.7, 0) = 0$

Thus, $R(x_1, x_1) \geq i(R(x, y), R(y, x_1))$

Now,

$R(x_1, x_2) = 0$

$i(R(x_1, x_1), R(x_1, x_2)) = i(1, 0) = 0$

$i(R(x_1, x_2), R(x_2, x_2)) = i(0, 0.2) = 0$

$i(R(x_1, x_3), R(x_3, x_2)) = i(0.7, 0.5) = 0.35$

$R(x_1, x_2) \not\geq i(R(x_1, y), R(y, x_2))$ for i = product.

$\therefore$ R is not i- transitive

* **Theorem :-**

A fuzzy relation R on X^2 is i- transitive iff $R \overset{i}{\circ} R \subseteq R$ (i.e, $R^2 \subseteq R$)

proof:-

let R be an i- transitive fuzzy relation on X^2

By def^n,

$R(x, z) \geq i(R(x, y), R(y, z))$, $\forall x, y \in X$ — ①

Now, for any $(x, z) \in X^2$

$$(R \overset{i}{\circ} R)(x, z) = \sup_{y \in Y} i(R(x, y), R(y, z))$$

$$\leq \sup_{y \in Y} R(x, z) \quad (\because \text{by eq}^n ①)$$

$$(R \overset{i}{\circ} R)(x, z) \leq R(x, z)$$

$$R \overset{i}{\circ} R \subseteq R$$

Conversely,

suppose, $R \overset{i}{\circ} R \subseteq R$ holds

$\Rightarrow (R \overset{i}{\circ} R)(x,z) \leq R(x,z)$, $\forall (x,z) \in X^2$

$\Rightarrow \sup_{y \in Y} i(R(x,y), R(y,z)) \leq R(x,z)$

$\Rightarrow R(x,z) \geq i(R(x,y), R(y,z))$, $\forall x,y,z \in X$

$\Rightarrow R$ is i- transitive.

* i- transitive closure :-

If a fuzzy relation R on X^2 is not i- transitive. We define i- transitive closure as the smallest i-transitive relation containing R. The i- transitive closure of R is denoted by $R_{T(i)}$.

- Remark :-

We introduce the following notation for composition power of R.

i.e., $R = R^{(1)}$

$R \overset{i}{\circ} R = R^{(2)}$

$R \overset{i}{\circ} R^2 = R^{(3)}$

$\vdots$

$R \overset{i}{\circ} R^{n-1} = R^{(n)}$

* Theorem :-

For any fuzzy relation R on X^2 the fuzzy relation given by $R_{T(i)} = \bigcup_{n=1}^{\infty} R^{(n)}$ is the i-transitive closure of R.

proof :-

Given that, $R_{T(i)} = \bigcup_{n=1}^{\infty} R^{(n)}$

First we show that, $R_{T(i)}$ is i- transitive

consider,

$$R_{T(i)} \dot{o} R_{T(i)} = \left(\bigcup_{n=1}^{\infty} R^{(n)}\right) \dot{o} \left(\bigcup_{m=1}^{\infty} R^{(m)}\right)$$

$$= \bigcup_{m=1}^{\infty}\left[\bigcup_{n=1}^{\infty} R^{(n)} \dot{o} R^{(m)}\right]$$

$$= \bigcup_{m=1}^{\infty} \bigcup_{n=1}^{\infty}\left[R^{(n)} \dot{o} R^{(m)}\right]$$

$$= \bigcup_{k=1}^{\infty} R^{(m+n)}$$

$$= \bigcup_{k=1}^{\infty} R^{k} \quad (\because k = mn)$$

$$= R_{T(i)}$$

i.e,

$$R_{T(i)} \dot{o} R_{T(i)} \subseteq R_{T(i)}$$

Hence, $R_{T(i)}$ is transitive

consider,

An i-transitive fuzzy relation 'S' & $R \subseteq S$ then

$R^2 = R \dot{o} R \subseteq S \dot{o} S \subseteq S$ $\quad (\because S$ is i-transitive$)$

$\Rightarrow R^2 \subseteq S$

$\|^{ly}$, $R^3 = R \dot{o} R^2 \subseteq S \dot{o} S^2 \subseteq S$

$\Rightarrow R^3 \subseteq S$

continuing this process.

$$R^n \subseteq S$$

$$\bigcup_{n=1}^{\infty} R^{(n)} \subseteq S$$

$$R_{T(i)} \subseteq S$$

This show that, $R_{T(i)}$ is the smallest fuzzy relation containing R.

$\therefore R_{T(i)}$ is the i-transitive closure of R.

Hence, the proof.

Ex.1) let $M = \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$

find i-transitive closure of M.
where, i) $i = \min$, ii) $i =$ product

<u>sol^n</u> :-

$i =$ product

i.e. $i(a,b) = ab$

then $(M \overset{i}{\circ} M)(x,y) = \sup_z i(M(x,z), M(z,y))$

$$= \sup_z [M(x,z) . M(z,y)]$$

then $(M \overset{i}{\circ} M) = M^{(2)} = \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0.35 & 0.7 \\ 0.3 & 0.04 & 0.21 \\ 0.15 & 0.45 & 1 \end{bmatrix}$$

$M \overset{i}{\circ} M^{(2)} = M^{(3)} = \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0.35 & 0.7 \\ 0.3 & 0.04 & 0.21 \\ 0.15 & 0.5 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0.35 & 0.7 \\ 0.3 & 0.105 & 0.21 \\ 0.15 & 0.5 & 1 \end{bmatrix}$$

$M_{T(i)} = M^{(1)} \cup M^{(2)} \cup M^{(3)} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

$$M_{T(i)} = M^{(1)} \cup M^{(2)} \cup M^{(3)} =$$

$$\begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0.37 & 0.7 \\ 0.3 & 0.04 & 0.21 \\ 0.15 & 0.5 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0.35 & 0.7 \\ 0.3 & 0.105 & 0.21 \\ 0.15 & 0.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.35 & 0.7 \\ 0.3 & 0.2 & 0.2 \\ 0.15 & 0.5 & 1 \end{bmatrix}$$

$M_{T(i)}$ is the i-transitive closure of a given fuzzy relation M.

i) $i = \min$.

$$(M \overset{i}{\circ} M)(x,y) = \sup_z i\,(M(x,z), M(z,y))$$

$$= \sup_z \min(M(x,z), M(z,y))$$

$$M \overset{i}{\circ} M = M^{(2)} = \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.35 & 0.7 \\ 0.6 & 0.04 & 0.35 \\ 0.15 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.3 & 0.2 & 0.3 \\ 0.3 & 0.5 & 1 \end{bmatrix}$$

$$M \overset{i}{\circ} M^{(2)} = M^{(3)} = \begin{bmatrix} 1 & 0 & 0.7 \\ 0.3 & 0.2 & 0 \\ 0 & 0.5 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.3 & 0.2 & 0.3 \\ 0.3 & 0.5 & 1 \end{bmatrix}$$

$$M(3) = \begin{bmatrix} 1 & \overset{0.5}{\cancel{0.5}} & 0.7 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.5 & 1 \end{bmatrix}$$

$$M_{T(i)} = \begin{bmatrix} 1 & 0.5 & 0.7 \\ 0.3 & 0.3 & 0.3 \\ 0.3 & 0.5 & 1 \end{bmatrix}$$

2) let $R = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix}$

be a matrix representation of a fuzzy relation R on a set X. find i- transitive closure of R where (i) i= min (ii) i= product.

⇒ i= product

~~then~~, $(R \overset{i}{\circ} R)(x,y) = \sup_z i\,(M(x,z), M(z,y))$

$$= \sup_z [M(x,z), M(z,y)]$$

then $(R \overset{i}{\circ} R) = R^{(2)} = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix}$

$$R(2) = \begin{bmatrix} 1 & 0.3 & 0.7 \\ \overset{0.12}{0.2} & 1 & 0.4 \\ 0.3 & 0.09 & 1 \end{bmatrix}$$

$$R(3) = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0.09 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0.09 & 1 \end{bmatrix}$$

$R_{T(i)} = R^{(1)} \cup R^{(2)} \cup R^{(3)} =$

$$\begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0.09 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0.09 & 1 \end{bmatrix}$$

$$R_{T(i)} = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0.09 & 1 \end{bmatrix}$$

$R_{T(i)}$ is the i- transitive closure of a given fuzzy relation M.

i) i = min

$$(R \overset{i}{\circ} R)(x, y) = \sup_{z} i\,(R(x, z), R(z, y))$$

$$= \sup_{z} \min\,(R(x, z), R(z, y))$$

$$R \overset{i}{\circ} R = R^{(2)} = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.3 & 1 & 0.4 \\ 0.3 & 0.3 & 1 \end{bmatrix}$$

$$R^{(3)} = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.2 & 1 & 0.4 \\ 0.3 & 0 & 1 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.3 & 1 & 0.4 \\ 0.3 & 0.3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.3 & 1 & 0.4 \\ 0.3 & 0.3 & 1 \end{bmatrix}$$

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$$R_{T(i)} = \begin{bmatrix} 1 & 0.3 & 0.7 \\ 0.3 & 1 & 0.4 \\ 0.3 & 0.3 & 1 \end{bmatrix}$$

* Inf - ω_i composition of fuzzy relation :-

For a continuous t-norm 'i' we define ω_i operation on $[0,1] \times [0,1]$ by

$\omega_i(a,b) = \sup\{x / x \in [0,1]$ such that $i(a,x) \le b\}$ $\forall\ a,b \in [0,1]$

This binary operation is called an ω_i operation where i is a t-norm

This ω_i operation represents logical implication.

e.g.

1) If $i = \min$ then

$\omega_i(a,b) = \sup\{x \mid \min(a,x) \le b\}$

$$\omega_i(a,b) = \begin{cases} 1 & a \le b \\ b & a > b \end{cases}$$

2) If $i =$ product then

$\omega_i(a,b) = \sup\{x / i(a,x) / b\}$

$= \sup\{x / ax \le b\}$

$$\omega_i(a,b) = \begin{cases} \frac{b}{a} & b < a \\ 1 & a \le b \end{cases}$$

*) Theorem :-

<u>Statement</u> :- For any $a, b, d \in [0,1]$ the binary operation ω_i has the following properties.

① $i(a,b) \leq d$ iff $\omega_i(a,d) \geq b$

② $\omega_i(\omega_i(a,b), b) \geq a$

③ $\omega_i(i(a,b), d) = \omega_i(a, \omega_i(b,d))$

proof

1) let $a, b, d \in [0,1]$

then $i(a,b) \leq d$

$\Rightarrow b \in \{x \mid i(a,x) \leq d\}$

$\Rightarrow b \leq \omega_i(a,d)$

i.e, $\omega_i(a,d) \geq b$

Conversely

if $\omega_i(a,d) \geq b$

$\Rightarrow i(a, \omega_i(a,d)) \geq i(a,b)$ by monotonocity

$\Rightarrow i(a, \sup\{x \mid i(a,x) \leq d\}) \geq i(a,b)$

$\Rightarrow \sup\{i(a,x) \mid i(a,x) \leq d\} \geq i(a,b)$

$\Rightarrow d \geq i(a,b)$

i.e, $i(a,b) \leq d$

Thus, $i(a,b) \leq d$ iff $\omega_i(a,d) \geq b$

2) For $a, b \in [0,1]$

$\omega_i(a,b) \geq \omega_i(a,b)$ (by property ①)

$\Rightarrow i(a, \omega_i(a,b)) \leq b$ i is commutative

$i(\omega_i(a,b), a) \leq b$

$\Rightarrow \omega_i(\omega_i(a,b), b) \geq a$ :: by ①

③ By def^n of ω_i,

$\omega_i(a,b) = \sup\{x \mid i(a,x) \leq b\}$, for $a,b,d,x \in [0,1]$

consider,

$i(a,x) \leq \omega_i(b,d)$

$\Leftrightarrow \omega_i(b,d) \geq i(a,x)$

$\Leftrightarrow i(b, i(a,x)) \leq d$ by prop ①

$\Leftrightarrow i(i(b,a),x) \leq d$

$\Leftrightarrow i(i(a,b),x) \leq d$

$\Leftrightarrow \omega_i(i(a,b),d) \geq x$ by prop ①

Thus, $i(a,x) \leq \omega_i(b,d)$

$\Leftrightarrow \omega_i(i(a,b),d) \geq x$

Next consider,

$\omega_i(a, \omega_i(b,d)) = \sup\{x \mid i(a,x) \leq \omega_i(b,d)\}$

$= \sup\{x \mid \omega_i(i(a,b),d) \geq x\}$

$= \omega_i(i(a,b),d)$

$\omega_i(a, \omega_i(b,d)) = \omega_i(i(a,b),d)$

Hence the proof.

* Theorem

For any $a,b,d \in [0,1]$

The operation ω_i has the following properties

④ $a \leq b \Rightarrow \omega_i(a,d) \geq \omega_i(b,d)$

& $\omega_i(d,a) \leq \omega_i(d,b)$

⑤ $i(\omega_i(a,b), \omega_i(b,d)) \leq \omega_i(a,d)$

⑥ $i(a, \omega_i(a,b)) \leq b$.

proof :-

4) let $a,b,d \in [0,1]$ & $a \leq b$

by def^n

$\omega_i(b,d) = \sup\{x \mid i(b,x) \leq d\}$,

but $a \leq b$

Now,

$\{x \mid i(b,x) \le d\} \subseteq \{x \mid i(a,x) \le d\}$

$\Rightarrow \sup\{x \mid i(b,x) \le d\} \le \sup\{x \mid i(a,x) \le d\}$

$\Rightarrow \omega_i(b,d) \le \omega_i(a,d)$

i.e. $\omega_i(a,d) \ge \omega_i(b,d)$

Now,

$\omega_i(d,a) = \sup\{x \mid i(d,x) \le a\}$

$\omega_i(d,b) = \sup\{x \mid i(d,x) \le b\}$

Now, $a \le b$

Hence,

$\{x \mid i(d,x) \le a\} \subseteq \{x \mid i(d,x) \le b\}$

$\{x \mid i(d,x) \le a\} \le \sup\{x \mid i(d,x) \le b\}$

$\Rightarrow \sup\{x \mid i(d,x) \le a\}$

$\Rightarrow \omega_i(d,a) \le \omega_i(d,b)$

5) we have, $a \le b$

$\omega_i(a,d) \ge \omega_i(b,d)$ ∵ (by property 4)

∴ for $a, b \in [0,1]$

$\omega_i(a,b) \ge \omega_i(a,b)$

$\Leftrightarrow i(a, \omega_i(a,b)) \le b$

$\Leftrightarrow i(\omega_i(a,b), a) \le b$

$\Leftrightarrow \omega_i(i(\omega_i(a,b), d), d) \ge \omega_i(b,d)$

$\Leftrightarrow \omega_i(\omega_i(a,b), \omega_i(a,d)) \ge \omega_i(b,d)$ by ①

$\Leftrightarrow i(\omega_i(a,b), \omega_i(b,d)) \le \omega_i(a,d)$ by ①

6) We have the following property.

$\omega_i(\omega_i(a,b), b) \ge a$, $\forall a, b \in [0,1]$

$i(\omega_i(a,b), a) \le b$ (by pro ②)

$\Rightarrow i(a, \omega_i(a,b)) \le b$ (i is commutative)

Hence the proof.

* ~~Prove~~ Theorem :-

For any $a, b, a_j \in [0,1]$, $j \in J$ the ω_j operation has the following properties.

⑦ $\omega_i\left(\inf_{j \in J} a_j, b\right) \geq \sup_{j \in J} \omega_i(a_j, b)$

⑧ $\omega_i\left(\sup_{j} a_j, b\right) = \inf_{j} \omega_i(a_j, b)$

⑨ $\omega_i\left(b, \sup_{j} a_j\right) \geq \sup_{j} \omega_i(b, a_j)$

⑩ $\omega_i\left(b, \inf_{j} a_j\right) = \inf_{j} \omega_i(b, a_j)$

proof :-

⑦ We have.

$\inf_{j} a_j \leq a_j \quad , \quad \forall j \in J$

$\omega_i\left(\inf_{j} a_j, b\right) \geq \omega_i(a_j, b)$ (by pro ⑥)

$\Rightarrow \omega_i\left(\inf_{j} a_j, b\right) \geq \sup_{j} \omega_i(a_j, b)$

⑧ We have $\sup_{j} a_j \geq a_j$

$\Rightarrow \omega_i\left(\sup_{j} a_j, b\right) \leq \omega_i(a_j, b) \quad \forall j \in J$

$\Rightarrow \omega_i\left(\sup_{j} a_j, b\right) \leq \inf_{j} \omega_i(a_j, b)$ — ①

Next,

$\omega_i(a_j, b) \leq \omega_i(a_j, b) \quad , \forall j \in J$

$\Rightarrow \inf_j \omega i(a_j, b) \le \omega i(a_j, b)$

$\Rightarrow i(a_j, \inf_j \omega i(a_j, b)) \le b$

$\Rightarrow \sup_j i(a_j, \inf_j \omega i(a_j, b)) \le b$

$\Rightarrow i(\sup_j a_j, \inf_j \omega i(a_j, b)) \le b$

let, $\sup_j a_j = s$

$i(s, \inf_j \omega i(a_j, b)) \le b$

$\Rightarrow \omega i(s, b) \ge \inf_j \omega i(a_j, b)$

$\Rightarrow \omega i(\sup_j a_j, b) \ge \inf_j \omega i(a_j, b)$ —— ②

By ① & ②,

$\omega i(\sup_j a_j, b) = \inf_i \omega i(a_j, b)$

② We have,

$\sup_j a_j \ge a_j$

$\omega i(b, \sup_j a_j) \ge \omega i(b, a_j) \ \forall j$

$\Rightarrow \omega i(b, \sup_i a_j) \ge \sup_j \omega i(b, a_j)$.

(10) We have,

$$\inf_j a_j \le a_j \quad , \quad \forall j$$

$$\Rightarrow \omega_i\left(b, \inf_j a_j\right) \le \omega_i(b, a_j)$$

$$\Rightarrow \omega_i\left(b, \inf_j a_j\right) \le \inf_j \omega_i(b, a_j) \quad \text{— (3)}$$

Next,

$$\omega_i(b, a_j) \le \omega_i(b, a_j) \quad , \quad \forall j$$

$$\inf_j \omega_i(b, a_j) \le \omega_i(b, a_j)$$

$$\Rightarrow i\left(b, \inf_j \omega_i(b, a_j)\right) \le a_j$$

$$\Rightarrow i\left(b, \inf_j \omega_i(b, a_j)\right) \le \inf a_j$$

$$\Rightarrow \omega_i\left(b, \inf_j a_j\right) \ge \inf_j \omega_i(b, a_j) \quad \text{— (4)}$$

from (3) & (4),

$$\omega_i\left(b, \inf_j a_j\right) = \inf_j \omega_i(b, a_j)$$

Hence the proof.

* Def^n:-

Inf ω_i composition of fuzzy relation

let i be t-norm & ω_i be a binary operation on $[0,1]$.

let $P(X,Y)$ & $Q(Y,Z)$ be a fuzzy relation

The infimum - ω_i composition P & Q is defined by,

$$(P \overset{\omega_i}{\circ} Q)(x,z) = \inf_{y \in Y} \omega_i\left(P(x,y), Q(y,z)\right) \quad , \quad \forall (x,z) \in X \times Z$$

** Theorem
let $P(X,Y)$, $Q(Y,Z)$, $R(X,Z)$ & $S(Z,V)$ be the fuzzy relation then following properties are equivalent

① (i) $P \overset{i}{\circ} Q \subseteq R$

(ii) $Q \subseteq P^{-1} \overset{\omega_i}{\circ} R$

(iii) $P \subseteq (Q \overset{\omega_i}{\circ} R^{-1})^{-1}$

Imp

② $P \overset{\omega_i}{\circ} (Q \overset{\omega_i}{\circ} S) = (P \overset{i}{\circ} Q) \overset{\omega_i}{\circ} S$

Solution:-
① To prove the equivalence, we show that
(i) ⇔ (ii) & (i) ⇔ (iii)
First we show that (i) ⇔ (ii)
consider,

$\cancel{P \overset{i}{\circ} Q \leq R}$

$\Leftrightarrow (P \overset{i}{\circ} Q)(x,z) \leq R(x,z)$

$\Leftrightarrow \sup_{y \in Y} i(P(x,y), Q(y,z)) \leq R(x,z) \quad \forall (x,z) \in X \times Z$

$\Leftrightarrow i(P(x,y), Q(y,z)) \leq R(x,z) \quad \forall (x,z) \in X \times Z$

$\Leftrightarrow \omega_i(P(x,y), R(x,z)) \geq Q(y,z) \quad \forall x \in X, y \in Y, z \in Z$

($\because$ by property ①)

$\Leftrightarrow \omega_i(P^{-1}(y,x), R(x,z)) \geq Q(y,z)$

$\Leftrightarrow \inf_{x \in X} \omega_i(P^{-1}(y,x), R(x,z)) \geq Q(y,z)$

$(P^{-1} \overset{\omega_i}{\circ} R)(y,z) \geq Q(y,z)$

i.e., $Q \leq P^{-1} \overset{\omega_i}{\circ} R$

Next we show that, (i) $\Leftrightarrow$ (iii)

Now $P \overset{i}{\circ} Q \subseteq R$

$\Leftrightarrow (P \overset{i}{\circ} Q)(x,z) \le R(x,z), \quad \forall (x,z) \in X \times Z$

$\Leftrightarrow \sup_{y \in Y} i\big(P(x,y), Q(y,z)\big) \le R(x,z), \quad \forall x \in X, y \in Y, z \in Z$

$\Leftrightarrow i\big(P(x,y), Q(y,z)\big) \le R(x,z), \quad \forall x \in X, y \in Y, z \in Z$

$\Leftrightarrow i\big(Q(y,z), P(x,y)\big) \le R(x,z), \quad \forall x \in X, y \in Y, z \in Z$

$\because$ i is commutative.

$\Leftrightarrow \omega_i\big(Q(y,z), R(x,z)\big) \ge P(x,y) \quad \forall x \in X, y \in Y, z \in Z$

by pro. (i)

($\because i(a,b) \le d$ iff $\omega_i(a,d) \ge b$)

$\Leftrightarrow \omega_i\big(Q(y,z), R^{-1}(z,x)\big) \ge P(x,y)$

$\Leftrightarrow \inf_{z \in Z} \omega_i\big(Q(y,z), R^{-1}(z,x)\big) \ge P(x,y) \quad \forall x \in X, y \in Y, z \in Z.$

$\Leftrightarrow (Q \overset{\omega_i}{\circ} R^{-1})(y,x) \ge P(x,y)$

$\Leftrightarrow (Q \overset{\omega_i}{\circ} R^{-1})^{-1}(x,y) \ge P(x,y)$

$\Leftrightarrow (Q \overset{\omega_i}{\circ} R^{-1})^{-1} \supseteq P$

i.e. $P \subseteq (Q \overset{\omega_i}{\circ} R^{-1})^{-1}$

(2) For any $x \in X$, $v \in V$

$$[(P \overset{\omega_i}{\circ} (Q \overset{\omega_i}{\circ} S)](x,v) = \inf_{y \in Y} \omega_i\big(P(x,y), (Q \overset{\omega_i}{\circ} S)(y,v)\big)$$

$$= \inf_{y \in Y} \omega i\left(P(x,y), \inf_{z \in Z} \omega i(Q(y,z), S(z,v))\right)$$

$$= \inf_{y \in Y} \inf_{z \in Z} \omega i\left(P(x,y), \omega i(Q(y,z), S(z,v))\right)$$

$$= \inf_{y \in Y} \inf_{z \in Z} \omega i\left(i(P(x,y), Q(y,z)), S(z,v)\right)$$

$(\omega i(i(a,b), d) = \omega i(a, \omega i(b,d)) \Rightarrow$ by pro ③ $)$

$$= \inf_{z \in Z} \omega i\left(\sup_{y \in Y} i(P(x,y), Q(y,z)), S(z,v)\right)$$

by ②

$\left[\because \omega i(\sup a_j, b) = \inf_j \omega i(a_j, b) \right]$

$$= \inf_{z \in Z} \omega i((P \overset{i}{\circ} Q)(x,z), S(z,v))$$

$$= \left[(P \overset{i}{\circ} Q) \overset{\omega i}{\circ} S \right] (x,v).$$

$$\therefore P \overset{\omega i}{\circ} (Q \overset{\omega i}{\circ} S) = (P \overset{i}{\circ} Q) \overset{\omega i}{\circ} S$$

Hence the proof.

** Theorem 8:-

Imp

let $P(X,Y)$, $P_j(X,Y)$, $Q(Y,Z)$, $Q_j(Y,Z)$, $j \in J$ be fuzzy relation then

① $\left(\bigcup_{j \in J} P_j \right) \overset{\omega i}{\circ} Q = \bigcap_j (P_j \overset{\omega i}{\circ} Q)$

② $\left(\bigcap_j P_j \right) \overset{\omega i}{\circ} Q \supseteq \bigcup_j (P_j \overset{\omega i}{\circ} Q)$

③ $P \overset{\omega i}{\circ} \left(\bigcap_j Q_j \right) = \bigcap_j (P \overset{\omega i}{\circ} Q_j)$

④ $P \overset{\omega_i}{\circ} (\bigcup_j Q_j) = \bigcup_j (P \overset{\omega_i}{\circ} Q_j)$

⇒ ① let $(x, z) \in X \times Z$ be arbitrary then.
consider,

$$\left[\left(\bigcup_{j \in J} P_j\right) \overset{\omega_i}{\circ} Q\right](x, z) = \inf_{y \in Y} \omega_i\left(\left(\bigcup_j P_j\right)(x, y), Q(y, z)\right)$$

$$= \inf_{y \in Y} \omega_i\left(\sup_j P_j(x, y), Q(y, z)\right)$$

$$= \inf_{y \in Y} \inf_j \omega_i\left(P_j(x, y), Q(y, z)\right)$$

(∵ by ⑧)

$$= \inf_j \inf_{y \in Y} \omega_i\left(P_j(x, y), Q(y, z)\right)$$

$$= \inf_j \left(P_j \overset{\omega_i}{\circ} Q\right)(x, z)$$

$$= \bigcap_j \left(P_j \overset{\omega_i}{\circ} Q\right)(x, z)$$

Thus, $\left(\bigcup_{j \in J} P_j\right) \overset{\omega_i}{\circ} Q = \bigcap_j \left(P_j \overset{\omega_i}{\circ} Q\right)$

② let $(x, z) \in X \times Z$
consider,

$$\left[\left(\bigcap_j P_j\right) \overset{\omega_i}{\circ} Q\right](x, z) = \inf_{y \in Y} \omega_i\left(\left(\bigcap_j P_j\right)(x, y), Q(y, z)\right)$$

$$= \inf_{y \in Y} \omega_i\left(\inf_j P_j(x, y), Q(y, z)\right)$$

$$\geq \inf_{y \in Y} \sup_{j \in J} \omega_i\left(P_j(x, y), Q(y, z)\right)$$

$$\geq \sup_j \inf_{y \in Y} \omega_i\left(P_j(x, y), Q(y, z)\right)$$

$$\geq \sup_j \; \cancel{\{P_j(x,y)} \; (P_j \overset{\omega_i}{\circ} Q)(x,z)$$

$$\geq \bigcup_j (P_j \overset{\omega_i}{\circ} Q)(x,z)$$

i.e, $\left(\bigcap_j P_j\right) \overset{\omega_i}{\circ} Q \supseteq \bigcup_j (P_j \overset{\omega_i}{\circ} Q)$

③ let $(x,z) \in X \times Z$
consider,

$$P \overset{\omega_i}{\circ} \left(\bigcap_j Q_j\right)(x,z) = \inf_{y \in Y} \omega_i\left(P(x,y), \left(\bigcap_j Q_j\right)(y,z)\right)$$

$$= \inf_{y \in Y} \omega_i\left(P(x,y), \inf_j Q_j(y,z)\right)$$

$$= \inf_{y \in Y} \inf_j \omega_i\left(P(x,y), Q_j(y,z)\right) \quad (\because \text{by } ⑬)$$

$$= \inf_j \left(P \overset{\omega_i}{\circ} Q_j\right)(x,z)$$

$$= \bigcap_j \left(P \overset{\omega_i}{\circ} Q_j\right)(x,z)$$

Thus, $P \overset{\omega_i}{\circ} \left(\bigcap_j Q_j\right) = \bigcap_j \left(P \overset{\omega_i}{\circ} Q_j\right)$

④ let $(x,z) \in X \times Z$
consider,

$$P \overset{\omega_i}{\circ} \left(\bigcup_j Q_j\right)(x,z) = \inf_{y \in Y} \omega_i\left(P(x,y), \bigcup_j Q_j(y,z)\right)$$

$$= \inf_{y \in Y} \omega_i\left(P(x,y), \sup_j Q_j(y,z)\right)$$

$$\geq \inf_{y \in Y} \sup_j \omega_i(P(x,y), Q_j(y,z))$$

($\because$ by ⑨)

$$\geq \sup_j (P \overset{\omega_i}{\circ} Q_j)(x,z)$$

$$\geq \bigcup_j (P_j \overset{\omega_i}{\circ} Q)(x,z)$$

i.e, $P \overset{\omega_i}{\circ} (\bigcup_j Q_j) \supseteq \bigcup_j (P \overset{\omega_i}{\circ} Q_j)$

Hence the proof.

* Theorem:-

let $P(X,Y)$, $Q_1(Y,Z)$, $Q(Y,Z)$, & $R(Z,V)$ be the fuzzy relations. if $Q_1 \subseteq Q_2$ then

① $P \overset{\omega_i}{\circ} Q_1 \subseteq P \overset{\omega_i}{\circ} Q_2$

② $Q_1 \overset{\omega_i}{\circ} R \supseteq Q_2 \overset{\omega_i}{\circ} R$

→

If $Q_1 \subseteq Q_2$ then

$Q_1 \cap Q_2 = Q_1$ & $Q_1 \cup Q_2 = Q_2$

also we have,

$A \cap B = A \Rightarrow A \subseteq B$

& $A \cap B = B \Rightarrow$ ~~A⊆B~~ $B \subseteq A$

where, A & B are fuzzy sets

$\therefore$ by using previous theorem,

① $(P \overset{\omega_i}{\circ} Q_1) \cap (P \overset{\omega_i}{\circ} Q_2) = P \overset{\omega_i}{\circ} (Q_1 \cap Q_2)$

$= P \overset{\omega_i}{\circ} Q_1$ $\{\because Q_1 \cap Q_2 = Q_1\}$

$\Rightarrow (P \overset{\omega_i}{\circ} Q_1) \subseteq (P \overset{\omega_i}{\circ} Q_2)$

② consider,

$$(Q_1 \overset{\omega i}{o} R) \cap (Q_2 \overset{\omega i}{o} R) = (Q_1 \cup Q_2) \overset{\omega i}{o} R$$ (above the cond^n ①)

$$= Q_2 \overset{\omega i}{o} R \quad \{\because Q_1 \cup Q_2 = Q_2,\ Q_1 \subseteq Q_2\}$$

$\Rightarrow (Q_2 \overset{\omega i}{o} R) \subseteq Q_1 \overset{\omega i}{o} R$

i.e. $Q_1 \overset{\omega i}{o} R \supseteq Q_2 \overset{\omega i}{o} R$.

Hence the proof.

* * Theorem :-

Imp

let $P(X,Y)$, $Q(Y,Z)$, & $R(X,Z)$ be the fuzzy relations then

(i) $P^{-1} \overset{i}{o} (P \overset{\omega i}{o} Q) \subseteq Q$

(ii) $R \subseteq P \overset{\omega i}{o} (P^{-1} \overset{i}{o} R)$

(iii) $P \subseteq (P \overset{\omega i}{o} Q) \overset{\omega i}{o} Q^{-1}$

(iv) $R \subseteq (R \overset{\omega i}{o} Q^{-1}) \overset{\omega i}{o} Q$.

→

proof :

① $P \overset{\omega i}{o} Q \subseteq P \overset{\omega i}{o} Q$

$\Leftrightarrow (P \overset{\omega i}{o} Q) \subseteq ((P^{-1})^{-1} \overset{\omega i}{o} Q)$

$\Leftrightarrow (P^{-1}) \overset{i}{o} (P \overset{\omega i}{o} Q) \subseteq Q$ $\quad \{Q \subseteq P^{-1} \overset{\omega i}{o} R \Leftrightarrow P \overset{i}{o} Q \subseteq R\}$

Thus, $P^{-1} \overset{i}{o} (P \overset{\omega i}{o} Q) \subseteq Q \quad \forall P(X,Y)$ & $Q(Y,Z)$

(ii) for $P(X,Y)$ & $R(X,Z)$

$(P^{-1} \overset{i}{o} R) \subseteq (P^{-1} \overset{i}{o} R)$

$\Leftrightarrow R \subseteq (P^{-1})^{-1} \overset{\omega i}{o} (P^{-1} \overset{i}{o} R)$ $\quad \{P \overset{i}{o} Q \subseteq R \Leftrightarrow Q \subseteq P^{-1} \overset{\omega i}{o} R\}$

$\Leftrightarrow R \subseteq P \overset{\omega i}{o} (P^{-1} \overset{i}{o} R) \quad \forall P(X,Y)$ & $R(X,Z)$

(iii) by (1), $P^{-1} \dot{\circ} (P \overset{\omega_i}{\circ} Q) \subseteq Q$

$\Leftrightarrow [P^{-1} \dot{\circ} (P \overset{\omega_i}{\circ} Q)]^{-1} \subseteq Q^{-1}$

$\Leftrightarrow (P \overset{\omega_i}{\circ} Q)^{-1} \dot{\circ} P \subseteq Q^{-1}$

$\Leftrightarrow P \subseteq [(P \overset{\omega_i}{\circ} Q)^{-1}]^{-1} \overset{\omega_i}{\circ} Q^{-1}$

$\Leftrightarrow P \subseteq (P \overset{\omega_i}{\circ} Q) \overset{\omega_i}{\circ} Q^{-1}$

$P \leq Q$
$P(x,y) \leq Q(x,y)$
$P^{-1}(y,x) \leq Q^{-1}(y,x)$
$P^{-1} \leq Q^{-1}$

$\{\because Q \subseteq P^{-1} \overset{\omega_i}{\circ} R \Leftrightarrow P \dot{\circ} Q \subseteq R\}$

(iv) since,

$P^{-1} \dot{\circ} (P \overset{\omega_i}{\circ} Q) \subseteq Q$

where,

$P^{-1}(Y,X)$, $(P \overset{\omega_i}{\circ} Q)(X,Z)$ & $Q(Y,Z)$ are fuzzy relation, replace P^{-1} by R^{-1} and Q by Q^{-1}

then we can write

$R^{-1} \dot{\circ} (R \overset{\omega_i}{\circ} Q^{-1}) \subseteq Q^{-1}$

$\Rightarrow (R^{-1} \dot{\circ} (R \overset{\omega_i}{\circ} Q^{-1}))^{-1} \subseteq (Q^{-1})^{-1}$

$\Rightarrow (R \overset{\omega_i}{\circ} Q^{-1})^{-1} \dot{\circ} R \subseteq Q$

$\left\{ \begin{array}{l} P \dot{\circ} Q \subseteq R \\ \Rightarrow Q \subseteq P^{-1} \overset{\omega_i}{\circ} R \end{array} \right\}$

$\Rightarrow R \subseteq (R \overset{\omega_i}{\circ} Q^{-1}) \overset{\omega_i}{\circ} Q$

Hence the proof.

* Fuzzy relation equation :-

~~Relational~~

- Relational equation based on max min composition

let $P(X,Y)$, $Q(Y,Z)$ & $R(X,Z)$ are fuzzy relation then maxmin composition of this relation is given by, $P \circ Q = R$

$$(P \circ Q)(x,z) = \max_{y \in Y} \min \{ P(x,y), Q(y,z) \}$$
$$= R(x,z)$$

If the set ~~ex~~ X, Y, Z are finite then

$(P \circ Q)(x_i, z_k) = R(x_i, z_k)$

we denote,

$$\left. \begin{aligned} P(x_i, y_j) &= p_{ij} \\ Q(y_j, z_k) &= q_{jk} \\ R(x_i, z_k) &= r_{ik} \end{aligned} \right\} \text{———} *$$

$(P \circ Q)(x_i, z_k) = R(x_i, z_k)$

$\Rightarrow \max_j \min (P(x_i, y_j), Q(y_j, z_k)) = R(x_i, z_k)$

$\Rightarrow \max_j \min (p_{ij}, q_{jk}) = r_{ik}$ (by *)

Now, P is unknown relation to be determined then we have to obtain matrix

$P = [p_{ij}]$

i.e, we have to determine the elements p_{ij} s.t.

$$\max_j \min (p_{ij}, q_{jk}) = r_{ik}$$

* Proposition :-

Let $P \circ Q = R$ be a fuzzy relational eq^n over a finite set X, Y & Z. ~~If~~

6 marks

If $\max\limits_j q_{jk} < \max\limits_i r_{ik}$,

for some k then $S(Q,R) = \phi$, (where $S(Q,R)$ is a solution set for the relation equation $P \circ Q = R$.

* Example

consider a fuzzy relation equation

$$P \circ \begin{bmatrix} 0.9 & 0.5 \\ 0.7 & 0.8 \\ 1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 1 \end{bmatrix}$$

→

let, $P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix}$

when, $P \circ Q = R$ gives,

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0.5 \\ 0.7 & 0.8 \\ 1 & 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.3 \\ 0.2 & 1 \end{bmatrix}$$

for k=1

$\max\limits_j \{q_{jk}\}$

$\max\limits_j \{q_{j1}\}$

$q_{11} = 0.9$, $q_{21} = 0.7$, $q_{31} = 1$

$\therefore \max\{q_{11}, q_{21}, q_{31}\} = \max\{0.9, 0.7, 1\}$
$= 1$

$\max\limits_j \{q_{j1}\} = 1$

$$\max_i \{r_{i1}\} = \max\{r_{11}, r_{21}\}$$
$$= \max\{0.6, 0.2\}$$
$$= 0.6$$
$$\therefore \max_i \{r_{i1}\} = 0.6$$

for $k=2$

$$\max_j \{q_{j2}\} = \max\{q_{12}, q_{22}, q_{32}\}$$
$$= \max\{0.5, 0.8, 0.4\}$$
$$= 0.8$$

$$\max_i \{r_{i2}\} = \max\{r_{12}, r_{22}\}$$
$$= \max\{0.3, 1\}$$
$$= 1$$

$$\max_j (q_{jk}) < \max_i (r_{ik})$$

$\therefore$ There is no solution for this relational eqn

* Note :-

1) If Q & R are given the relational equation $P \circ Q = R$ is satisfies by some relation P. we say that $P \in S(Q, R)$
where, $S(Q, R)$ is a solution set for the relation equation .; $P \circ Q = R$

2) If $P \circ Q = R$, where P & R are given then we have, $(P \circ Q)^{-1} = R^{-1}$

$$\Rightarrow Q^{-1} \circ P^{-1} = R^{-1}$$

where, P^{-1} & R^{-1} are ~~are~~ given

2) let R & Q are given fuzzy relation we determine the fuzzy relation such that $PoQ = R$.

let $S(Q,R) = \{P \mid PoQ = R\}$

Now if $P = [p_{ij}]$, $Q = [q_{jk}]$ & $R = [r_{ik}]$

then $PoQ = R \Rightarrow [p_{ij}] \circ [q_{jk}] = [r_{ik}]$

$\Rightarrow p_i \circ Q = r_i \quad \forall i \quad$ (i = no. of rows)

where, p_i & r_i are the row vector of P & R respectively

i.e. $p_i = \{p_{ij} \mid j \in J\}$

& $r_i = \{r_{ik} \mid k \in K\}$

Thus ~~First~~, to determine P we find the row vector p_i, $i \in I$ & combining this p_i's we obtain the matrix for P.

Thus the problem of finding the relation P is reduce to getting the p_i's where, $p_i \in S(Q, r_i) \ \forall i$.

This procedure is called the partitioning of the problem.

* Methods for Maximal Solution :-

let $PoQ = R$ be a fuzzy relational eqn under max min composition.

let $PoQ = r$, be the partitioning of this relation eqn.

where, P & r are row vector of P & R resp.

5

An element $\hat{p} \in S(Q,r)$ is called a maximal solution of the eq^n if for any $p \in S(Q,r)$

$$p \geq \hat{p} \Rightarrow p = \hat{p}$$

$$p \leq \hat{p}$$

& for any $p \in S(Q,r)$, $p \leq \hat{p}$

If $S(Q,r) \neq \phi$ then the maximal sol^n is given by $\hat{p} = \{\hat{p} / j \in J\}$

where, $\hat{p}_j = \min_{k \in K} \sigma(q_{jk}, r_k)$

&

$$\sigma(q_{jk}, r_k) = \begin{cases} r_k & \text{if } q_{jk} > r_k \\ 1 & \text{otherwise} \end{cases}$$

If $\hat{p}$ determined by this formula, does not satisfy $\hat{p} \circ Q = r$ then

$S(Q,r) = \phi$

If $\hat{p} \circ Q = r$ by this formula is satisfied then $S(Q,r) \neq \phi$ & $\hat{p}$ is maximal solution

① Ex. Obtain the maximal solution for the fuzzy relation equation based on max min composition

$$P \circ \begin{bmatrix} 0.9 & 0.6 & 1 \\ 0.8 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.6 & 0.5 \end{bmatrix}$$

$\Rightarrow$

let $\hat{p} = [\hat{p}_1 \; \hat{p}_2 \; \hat{p}_3]$

be the row vector

where, $\hat{p}_1 = \min_{k} \sigma(q_{1k}, r_k)$

$= \min \{ \sigma(q_{11}, r_1), \sigma(q_{12}, r_2), \sigma(q_{13}, r_3) \}$

$\sigma(q_{11}, r_1) = \sigma(0.9, 0.6) = 0.6$

$\sigma(q_{12}, r_2) = \sigma(0.6, 0.6) = 1$

$\sigma(q_{13}, r_3) = \sigma(1, 0.5) = 0.5$

$\hat{P}_1 = \min \{ 0.6, 1, 0.5 \}$

$= 0.5$

$\hat{P}_2 = \min\limits_k \sigma(q_{2k}, r_k)$

$= \min \{ \sigma(q_{21}, r_1), \sigma(q_{22}, r_2), \sigma(q_{23}, r_3) \}$

$\sigma(q_{21}, r_1) = \sigma(0.8, 0.6) = 0.6$

$\sigma(q_{22}, r_2) = \sigma(0.8, 0.6) = 0.6$

$\sigma(q_{23}, r_3) = \sigma(0.5, 0.6) = 1$

$\hat{P}_2 = \min \{ 0.6, 0.6, 1 \}$

$= 0.6$

$\hat{P}_3 = \min\limits_k \sigma(q_{3k}, r_k)$

$= \min \{ \sigma(q_{31}, r_3), \sigma(q_{32}, r_2), \sigma(q_{33}, r_3) \}$

$= \min \{ 1, 1, 0.5 \}$

$\hat{P}_3 = 0.5$

consider,

$$\hat{p} \circ \begin{bmatrix} 0.9 & 0.6 & 1 \\ 0.8 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.6 & 1 \\ 0.8 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.5 \end{bmatrix}$$

$$= r$$

i.e. $\hat{\beta} \circ Q = r$ & also $\hat{\beta}$ is maximal
Hence, $\hat{\beta} \in S(Q, r)$ &
solution.

Ex ② obtain the maximal solⁿ for the fuzzy relation eqⁿ based on max min composition.

6 marks

$$P \circ \begin{bmatrix} 0.1 & 0.4 & 0.5 & 0.1 \\ 0.9 & 0.7 & 0.2 & 0 \\ 0.8 & 1 & 0.5 & 0 \\ 0.1 & 0.3 & 0.6 & 0 \end{bmatrix} = [0.8 \quad 0.7 \quad 0.5 \quad 0]$$

⇒

let $\hat{\beta} = [\hat{P}_1 \quad \hat{P}_2 \quad \hat{P}_3 \quad \hat{P}_4]$

be the row vector

where, $\hat{P}_i = \min_k \sigma(a_{ik}, r_k)$

$= \min(\sigma(a_{11}, r_1), \sigma(a_{12}, r_2), \sigma(a_{13}, r_3), \sigma(a_{14}, r_4)\}$

$= \min\{1, 1, 1, 0\}$

$= 0$

$\hat{P}_2 = \min(\sigma(a_{21}, r_1), \sigma(a_{22}, r_2), \sigma(a_{23}, r_3), \sigma(a_{24}, r_4))$

$= \min\{0.8, 1, 1, 1\}$

$= 0.8$

$\hat{P}_3 = \min\{\sigma(a_{31}, r_1), \sigma(a_{32}, r_2), \sigma(a_{33}, r_3), \sigma(a_{34}, r_4)\}$

$= \min\{1, 0.7, 1, 1\}$

$= 0.7$

$\hat{P}_4 = \min\{\sigma(a_{41}, r_1), \sigma(a_{42}, r_2), \sigma(a_{43}, r_3), \sigma(a_{44}, r_4)\}$

$= \min\{1, 1, 0.5, 1\}$

$= 0.5$

consider,

$$\hat{P} \circ \begin{bmatrix} 0.1 & 0.4 & 0.5 & 0.1 \\ 0.9 & 0.7 & 0.2 & 0 \\ 0.8 & 1 & 0.5 & 0 \\ 1 & 0.3 & 0.6 & 0 \end{bmatrix}$$

$$= [0, 0.8, 0.7, 0.5] \circ \begin{bmatrix} 0.1 & 0.4 & 0.5 & 0.1 \\ 0.9 & 0.7 & 0.2 & 0 \\ 0.8 & 1 & 0.5 & 0 \\ 1 & 0.3 & 0.6 & 0 \end{bmatrix}$$

$$= [0.8, 0.7, 0.6, 0]$$
$$= r$$

i.e, $\hat{P} \circ Q = r$

Hence, $\hat{P} \in S(Q, r)$ & also $\hat{P}$ is maximal solution.

Method for minimal solution of $P \circ Q = r$:-

let $P \circ Q = r$ be a fuzzy relation eq^n where, ~~$P \circ Q$~~ P & r are row vector.

step I)

Arrange components of row vector are in decreasing order i.e, $r_1 \geq r_2 \geq r_3 \geq \ldots$

step II)

obtain the maximal sol^n $\hat{P}$ for the eq^n $P \circ Q = r$

let $\hat{P} = (\hat{P}_j \,/\, j \in J)$

step III)

If $\hat{P}_j = 0$ for some $j \in J$ then ~~et~~ eliminate the j^{th} row from the matrix for Q

Step IV)

If $r_K = 0$, for some K then eliminate K^{th} column from Q.

This procedure will reduce the size of the matrix.

Finally the minimal soln of this reduced equation can be extended by ~~intersecting~~ inserting the zero's at resp position.

~~This~~

Step V)

Let $P \circ Q = R$ be the reduce equation whose maximal solution is $\hat{P}$

Step VI)

For $K \in K$,

define, $J_K(\hat{P}) = \{ j \in J / \min(\hat{P}_j, q_{jK}) = r_K \}$

let, $J(\hat{P}) = \underset{K \in K}{\times} J_K(\hat{P})$

Step VII)

For any $\beta \in J(\hat{P})$

Determine the set, $K(\beta, j) = \{ K \in K / \beta_K = j \}$ $j \in J$

Step VIII)

For each $\beta \in J(\hat{P})$

define a row vector

$g(\beta) = [\, g_j(\beta) \,/\, j \in J \,]$

$$g_j(\beta) = \begin{cases} \max\limits_{K \in K(\beta, j)} r_K & \text{if } K(\beta, j) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

step IX)
For all $g(\beta)$'s select the minimal element by pairwise comparision then the resulting set of n-tuple is the minimal soln.
step X)
There may be more than one minimal soln
if $1\check{p}, 2\check{p}, 3\check{p}$ are three minimal solution then set of all soln is given by,

$$S(Q, R) = [1\check{p}, \hat{p}] \cup [2\check{p}, \hat{p}] \cup [3\check{p}, \hat{p}]$$

Obtain the minimal soln for $P \circ Q = r$
where,

$$Q = \begin{bmatrix} 0.1 & 0.4 & 0.5 & 0.1 \\ 0.9 & 0.7 & 0.2 & 0 \\ 0.8 & 1 & 0.5 & 0 \\ 0.1 & 0.2 & 0.6 & 0 \end{bmatrix} \quad \& \quad r = [0.8\ 0.7\ 0.5\ 0]$$

⇒

Note that, the elements of small r are arrange in decreasing order.
Also the maximal soln of this eqn given by,
let $\hat{\beta} = [\hat{p}_1\ \hat{p}_2\ \hat{p}_3\ \hat{p}_4]$
where, $\hat{p}_1 = \min \sigma(a_{1k}, r_k)$

$$= \min(\sigma(a_{11}, r_1), \sigma(a_{12}, r_2), \sigma(a_{13}, r_3), \sigma(a_{14}, r_4)\}$$

$$= \min\{1, 1, 1, 0\}$$

$$= 0$$

$$\hat{p}_2 = \min\{\sigma(a_{21}, r_1), \sigma(a_{22}, r_2), \sigma(a_{23}, r_3), \sigma(a_{24}, r_4)\}$$

$$= \min\{0.8, 1, 1, 1\}$$

$$= 0.8$$

$$\hat{p}_3 = \min\{\sigma(q_{31}, r_1), \sigma(q_{32}, r_2), \sigma(q_{33}, r_3), \sigma(q_{34}, r_4)\}$$

$$= \min\{1, 0.7, 1, 1\}$$

$$= 0.7$$

$$\hat{p}_4 = \min\{\sigma(q_{41}, r_1), \sigma(q_{42}, r_2), \sigma(q_{43}, r_3), \sigma(q_{44}, r_4)\}$$

$$= \min\{1, 1, 0.5, 1\}$$

$$= 0.5$$

consider,

$$\hat{p} \circ \begin{bmatrix} 0.1 & 0.4 & 0.5 & 0.1 \\ 0.9 & 0.7 & 0.2 & 0 \\ 0.8 & 1 & 0.5 & 0 \\ 0.1 & 0.2 & 0.6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.8 & 0.7 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 & 0.4 & 0.5 & 0.1 \\ 0.9 & 0.7 & 0.2 & 0 \\ 0.8 & 1 & 0.5 & 0 \\ 0.1 & 0.2 & 0.6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 & 0.7 & 0.5 & 0 \end{bmatrix}$$

i.e, $\hat{p} \circ Q = r$

Hence, $\hat{p} \in S(Q, r)$ & also $\hat{p}$ is maximal sol^n

since, $\hat{p}_1 = 0$

we eliminate first row of Q,

similarly $r_4 = 0$

hence we eliminate 4^{th} column of Q &

$\therefore$ reduce eq^n is given by,

$$[P_1\ P_2\ P_3] \circ \begin{bmatrix} 0.9 & 0.7 & 0.2 \\ 0.8 & 1 & 0.5 \\ 0.1 & 0.2 & 0.6 \end{bmatrix} = [0.8\ 0.7\ 0.5]$$

let $\hat{P} = [0.8 \quad 0.7 \quad 0.5]$
be the maximal soln
where $\hat{P}_1 = 0.8$, $\hat{P}_2 = 0.7$, $\hat{P}_3 = 0.5$

For each $K \in \{1, 2, 3\}$
$J_K(\hat{P}) = \{ j \in J / \min(\hat{P}_j, q_{jk}) = r_k$

For $K=1$
$J_1(\hat{P}) = \{ j \in J / \min_j (\hat{P}_j, q_{j1}) = r_1 \}$

Now, $\min(\hat{P}_1, q_{11}) = \min(0.8, 0.9) = 0.8$

$\min(\hat{P}_2, q_{21}) = \min(0.7, 0.8) = 0.7$
$\min(\hat{P}_3, q_{31}) = \min(0.5, 0.1) = 0.1$
$\therefore \quad J_1(\hat{P}) = \{ j \in J / \min(\hat{P}_j, q_{j1}) = 0.8 \}$

$\therefore \quad J_1(\hat{P}) = \{1\}$

for $K = 2$

$J_2(\hat{P}) = \{ j \in J / \min(\hat{P}_j, q_{j2}) = 0.7 \}$

$\min(\hat{P}_1, q_{12}) = \min(0.8, 0.7) = 0.7$
$\min(\hat{P}_2, q_{22}) = \min(0.7, 1) = 0.7$
$\min(\hat{P}_3, q_{32}) = \min(0.5, 0.2) = 0.2$

$J_2(\hat{P}) = \{1, 2\}$

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for $k = 3$

$J_3(\hat{\beta}) = \{ j \in J \,/\, \min(\hat{p}_j, q_{j3}) = 0.5 \}$

$\min(\hat{p}_1, q_{13}) = \min(0.8, 0.2) = 0.2$

$\min(\hat{p}_2, q_{23}) = \min(0.7, 0.5) = 0.5$

$\min(\hat{p}_3, q_{33}) = \min(0.5, 0.6) = 0.5$

$J_3(\hat{p}) = \{2, 3\}$

$$J(\hat{p}) = J_1(\hat{p}) \times J_2(\hat{p}) \times J_3(\hat{p})$$
$$= \{1\} \times \{1, 2\} \times \{2, 3\}$$
$$= \{(1,1,2), (1,1,3), (1,2,2)(1,2,3)\}$$

$$J(\hat{p}) = \{\underset{{}^1\beta}{(1,1,2)}\underset{{}^2\beta}{(1,2,2)}, \underset{{}^3\beta}{(1,1,3)}\underset{{}^4\beta}{(1,2,3)}\}$$

Define,

$K(\beta, j) = \{k \in K \,/\, \beta_k = j\}$

then for ${}^1\beta = (1,1,2)$, we have

$$K({}^1\beta, j) = \begin{cases} \{1, 2\} & j = 1 \\ \{3\} & j = 2 \\ \phi & j = 3 \end{cases}$$

i.e. $K({}^1\beta, 1) = \{1, 2\}$

$K({}^1\beta, 2) = \{3\}$

$K({}^1\beta, 3) = \phi$

For ${}^2\beta = (1,2,2)$

$$K({}^2\beta, j) = \begin{cases} \{1\} & j = 1 \\ \{2, 3\} & j = 2 \\ \phi & j = 3 \end{cases}$$

i.e, $K(2\beta, 1) = \{1\}$

$K(2\beta, 2) = \{2, 3\}$

$K(2\beta, 3) = \phi$

$3\beta = (1, 1, 3)$

$$K(3\beta, j) = \begin{cases} \{1, 2\} & j = 1 \\ \phi & j = 2 \\ \{3\} & j = 3 \end{cases}$$

i.e $K(3\beta, 1) = \{1, 2\}$

$K(3\beta, 2) = \phi$

$K(3\beta, 3) = \{3\}$

$4\beta = (1, 2, 3)$

$$K(4\beta, j) = \begin{cases} \{1\} & j = 1 \\ \{2\} & j = 2 \\ \{3\} & j = 3 \end{cases}$$

i.e, $K(4\beta, 1) = \{1\}$

$K(4\beta, 2) = \{2\}$

$K(4\beta, 3) = \{3\}$

define,

$g(\beta) = \{ g_j(\beta) / j \in J \}$

$g(\beta) = [g_1(\beta), g_2(\beta), g_3(\beta)]$

~~then~~ ~~for~~ we know that,

$$g_j(\beta) = \begin{cases} \max\limits_{k \in (\beta, j)} \pi_k & \text{if } K(\beta, j) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

then for $^1\beta$, we have

$g_1(^1\beta) = \max[0.8, 0.7] = 0.8$

$g_2(^1\beta) = \cancel{\max}\ 0.5$

$g_3(^1\beta) = 0$

$g(^1\beta) = [0.8, 0.5, 0]$

for 2β,

$g_1(^2\beta) = 0.8$

$g_2(^2\beta) = \max[0.7 . 0.5] = 0.7$

$g_3(^2\beta) = 0$

$g(2\beta) = [0.8, 0.7, 0]$

for 3β,

$g_1(^3\beta) = \max\{0.8 . 0.7\} = 0.8$

$g_2(^3\beta) = 0$

$g_3(^3\beta) = 0.5$

$g_3(^3\beta) = [0.8, 0, 0.5]$

$g_1(4\beta) = 0.8$

$g_2(4\beta) = 0.7$

$g_3(4\beta) = 0.5$

$g(4\beta) = [0.8, 0.7, 0.5]$

The sol^n set,

$\{g(1\beta), g(2\beta), g(3\beta), g(4\beta)\}$
$= \{[0.8, 0.5, 0], [0.8, 0.7, 0], [0.8, 0, 0.5]$
$[0.8, 0.7, 0.5]\}$

Thus the minimal are,

$\overset{1}{\check{p}} = [0.8, 0.5, 0]$
$\overset{2}{\check{p}} = [0.8, 0, 0.5]$

The sol^n set of reduce eq^n,
$S(Q, r) = [\,{}^1\check{p}, \hat{p}] \cup [\,{}^2\check{p}, \hat{p}]$

$\therefore$ sol^n set of original eq^n is
$S(Q, r) = [\,{}^1\check{p}, \hat{p}] \cup [\,{}^2\check{p}, \hat{p}]$
where,

$[\,{}^1\check{p}, \hat{p}] = [0.8, 0.5, 0]$
& $[\,{}^2\check{p}, \hat{p}] = [0.8, 0, 0.5]$

$\hat{p} = [0.8, 0.7, 0.5]$

Ex. Find the minimal sol^n of

$$P \circ \begin{bmatrix} 0.9 & 0.6 & 1 \\ 0.8 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.6 \end{bmatrix} = [0.6 \;\; 0.6 \;\; 0.5]$$

$\Rightarrow$ Note that, the elements of small r are arrange in decreasing order.
Also the maximal sol^n of this eq^n given by
let $\hat{p} = [\hat{p}_1 \; \hat{p}_2 \; \hat{p}_3 \; \hat{p}_4]$

where, $\hat{P}_1 = \min \sigma(a_{1K}, r_K)$

$= \min(\sigma(a_{11}, r_1), \sigma(a_{12}, r_2), \sigma(a_{13}, r_3))$

$= \min(0.6, 1, 1)$

$= \cancel{0.9}\ 0.6$

$= 0.6$

$\hat{P}_2 = \min \sigma(a_{2K}, r_K)$

$= \min(\sigma(a_{21}, r_1), \sigma(a_{22}, r_2), \sigma(a_{23}, r_3))$

$= \min(0.6, 0.6, 1)$

$= 0.6$

$= 0.6$

$\hat{P}_3 = \min(\sigma(a_{3K}, r_K))$

$= \min(\sigma(a_{31}, r_1), \sigma(a_{32}, r_2), \sigma(a_{33}, r_3))$

$= \min(1, 1, 0.5)$

$= 0.5$

$= 0.5$

consider,

$$\hat{P} \circ \begin{bmatrix} 0.9 & 0.6 & 1 \\ 0.8 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.6 & 1 \\ 0.8 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & 0.6 & 0.5 \end{bmatrix}$$

i.e, $\hat{P} \circ Q = r$

Hence, $\hat{P} \in S(Q, r)$ & also $\hat{P}$ is maximal sol^n.

$$[P_1\ P_2\ P_3] \circ \begin{bmatrix} 0.9 & 0.6 & 1 \\ 0.8 & 0.8 & 0.5 \\ 0.6 & 0.4 & 0.5 \end{bmatrix} = [0.6\ \ 0.6\ \ 0.5]$$

let $\hat{p} = [0.6\ \ 0.6\ \ 0.5]$

be the maximal solution,

where, $\hat{p}_1 = 0.6$, $\hat{p}_2 = 0.6$, $\hat{p}_3 = 0.5$

for each $k \in \{1,2,3\}$

$J_k(\hat{p}) = \{ j \in J / \min(\hat{p}_j, q_{jk}) = r_k$

for $k=1$

$J_1(\hat{p}) = \{ j \in J / \min\limits_j (\hat{p}_j, q_{j1}) = r_1 \}$

Now, $\min(\hat{p}_1, q_{11}) = \min(0.6, 0.9) = 0.6$

$\min(\hat{p}_2, q_{21}) = \min(0.6, 0.8) = 0.6$

$\min(\hat{p}_3, q_{31}) = \min(0.5, 0.6) = \cancel{0.6}\ 0.5$

$J_1(\hat{p}) = \{ j \in J / \min(\hat{p}_j, q_{j1}) = 0.6 \}$

$J_1(\hat{p}) = \{1,2\}$

for $k=2$

$J_2(\hat{p}) = \{ j \in J / \min\limits_j (\hat{p}_j, q_{j2}) = 0.6 \}$

now,

$\min(\hat{p}_1, q_{12}) = \min(0.6, 0.6) = 0.6$

$\min(\hat{p}_2, q_{22}) = \min(0.6, 0.8) = 0.6$

$\min(\hat{p}_3, q_{32}) = \min(0.5, 0.4) = 0.4$

$J_2(\hat{p}) = \{ j \in J / \min(\hat{p}_j, q_{j2}) = 0.6 \}$

$J_2(\hat{p}) = \{1,2\}$

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for $k=3$

$J_3(\hat{P}) = \{ j \in J \,/\, \min(\hat{P}_j, q_{j3}) = 0.5 \}$

$\min(\hat{P}_1, q_{13}) = \min(0.6, 1) = 0.6$

$\min(\hat{P}_2, q_{23}) = \min(0.6, 0.5) = 0.5$

$\min(\hat{P}_3, q_{33}) = \min(0.5, 0.6) = 0.5$

$J_3(\hat{P}) = \{2, 3\}$

$J(\hat{P}) = J_1(\hat{P}) \times J_2(\hat{P}) \times J_3(\hat{P})$

$= \{1,2\} \times \{1,2\} \times \{2,3\}$

$= (1,1,2)\ (1,1,3)\ (1,2,2)\ (1,2,3)$

$(2,1,2)\ (2,1,3)\ (2,1,2)\ (2,2,3)$

- Note :-

The values of $K(\beta, j)$ & $g(\beta)$ can be put in the following table.

β	$K(\beta,j)$ $j=1$	$j=2$	$j=3$	$g(\beta)$
(2,2,1)	{3}	{1,2}	ϕ	[0.5 0.6 0]
(2,2,2)	ϕ	{1,2,3}	ϕ	[0 0.6 0]
(2,2,3)	ϕ	{1,2}	{3}	[0 0.6 0.5]

* Equation based on sup-i composition.

let $P(X,Y)$, $Q(Y,Z)$ & $R(x,z)$ be the fuzzy relation. The eq^n of the time $P \overset{i}{\circ} Q = R$ is the relation equation. where Q & R are known and the relation P is unknown.

If $P=[p_{ij}]$, $Q=[q_{jk}]$ & $R=[r_{ik}]$ is the matrix representation of the fuzzy relations. then

then $P \overset{i}{\circ} Q = R$ gives $[p_{ij}] \circ [q_{jk}] = [r_{ik}]$

where, $\sup\limits_{j} i(p_{ij}, q_{jk}) = r_{ik}$

we denote, the set of all fuzzy relation P which satisfy $P \overset{i}{\circ} Q = R$ by $S(Q,R)$

i.e, $S(Q,R) = \{P / P \overset{i}{\circ} Q = R\}$.

* Theorem

If $S(Q,R) \neq \phi$ then the fuzzy relation $\hat{P} = (Q \overset{\omega_i}{\circ} R^{-1})^{-1}$ is the greatest member of $S(Q,R)$.

5 marks

Proof :- since, $S(Q,R) \neq \phi$

we assume that, $P_1 \in S(Q,R)$.
such that, $P_1 \dot{\circ} Q = R$
but $P \dot{\circ} Q = R$
consider, $P_1 \dot{\circ} Q \subseteq R$

$\Rightarrow P_1 \subseteq (Q \overset{\omega}{\circ}{}^{i} R^{-1})^{-1}$ $\quad (\because P \dot{\circ} Q \subseteq R \Leftrightarrow P \subseteq (Q \overset{\omega}{\circ}{}^{i} R^{-1})^{-1})$

$\Rightarrow P_1 \subseteq \hat{P}$ where $\hat{P} = (Q \overset{\omega}{\circ}{}^{i} R^{-1})^{-1}$

Thus, $\forall P_1 \in S(Q,R)$, $P_1 \subseteq \hat{P}$

we show that, $\hat{P}$ satisfies the equation $P \dot{\circ} Q = R$.
i.e. $\hat{P} \dot{\circ} Q = R$

let $T = \hat{P} \dot{\circ} Q$

$T = (Q \overset{\omega}{\circ}{}^{i} R^{-1})^{-1} \dot{\circ} Q$

$T^{-1} = [(Q \overset{\omega}{\circ}{}^{i} R^{-1})^{-1} \dot{\circ} Q]^{-1}$

$T^{-1} = Q^{-1} \dot{\circ} (Q \overset{\omega}{\circ}{}^{i} R^{-1})$ $\quad \{\because P^{-1} \dot{\circ} (Q \overset{\omega}{\circ}{}^{i} Q) \subseteq Q\}$

$T^{-1} \subseteq R^{-1}$

$\Rightarrow T \subseteq R$

$\Rightarrow \hat{P} \dot{\circ} Q \subseteq R$ —— (a)

Next for any $P_1 \in S(Q,R)$, we have;

$P_1 \subseteq \hat{P}$

$\Rightarrow P_1 \dot{\circ} Q \subseteq \hat{P} \dot{\circ} Q$

$\Rightarrow R \subseteq \hat{P} \dot{\circ} Q$ —— (b)

from eq^n (a) & (b)

$\hat{P} \dot{\circ} Q = R$

hence we have,

$\hat{P} \in S(Q,R)$

we shows that $\hat{P}$ is solution of the sup-i composition $P \dot{\circ} Q = R$ & also it is maximal sol^n of relation equation.

Thus $\hat{P} = (Q \overset{\omega}{\circ}{}^{i} R^{-1})^{-1}$ is the greatest member of $S(Q,R)$

Hence the proof.

Example

1) let $P \overset{i}{\circ} Q = R$ be the fuzzy relation equation where,

$$Q = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}, \quad R = \begin{bmatrix} 0.12 \\ 0.18 \\ 0.27 \end{bmatrix}$$

find the greatest solution of the eqn for
i) i = product.
ii) i = bounded difference.

Solution :-

i) $P \overset{i}{\circ} Q = R$ be the given equation.
where $Q = [0.1 \quad 0.2 \quad 0.3]^T$
& $R = [0.12 \quad 0.18 \quad 0.27]^T$

let $P = [p_{ij}]$, $Q = [q_{jk}]$, $R = [r_{jk}]$
by thm,

$$\hat{P} = (Q \overset{\omega_i}{\circ} R^{-1})^{-1}$$

is the greatest element of $S(Q, R)$
then $\hat{P}^{-1} = Q \overset{\omega_i}{\circ} R^{-1}$

$$\therefore \hat{P}^{-1} = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix} 0.12 & 0.18 & 0.27 \end{bmatrix}$$

$$= \begin{bmatrix} \omega_i(0.1, 0.12) & \omega_i(0.1, 0.18) & \omega_i(0.1, 0.27) \\ \omega_i(0.2, 0.12) & \omega_i(0.2, 0.18) & \omega_i(0.2, 0.27) \\ \omega_i(0.3, 0.12) & \omega_i(0.3, 0.18) & \omega_i(0.3, 0.27) \end{bmatrix}$$

for i = product

$$\omega_i(a, b) = \begin{cases} 1, & a \le b \\ \frac{b}{a}, & a > b \end{cases}$$

$$\hat{P}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0.6 & 0.9 & 1 \\ 0.4 & 0.6 & 0.9 \end{bmatrix}$$

Hence, the greatest soln is,

$$\hat{P} = \begin{bmatrix} 1 & 0.6 & 0.4 \\ 1 & 0.9 & 0.6 \\ 1 & 1 & 0.9 \end{bmatrix}$$

we verify that, $\hat{P} \in S(Q, R)$

consider, $\hat{P} \overset{i}{o} Q = \begin{bmatrix} 1 & 0.6 & 0.4 \\ 1 & 0.9 & 0.6 \\ 1 & 1 & 0.9 \end{bmatrix} \overset{i}{o} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$

$= \begin{bmatrix} 0.12 \\ 0.18 \\ 0.27 \end{bmatrix}$

$\hat{P} \overset{i}{o} Q = R$

Thus $\hat{P} \in S(Q, R)$ i.e., $S(Q, R) \neq \phi$.

$\therefore \hat{P}$ is the greatest soln of $P \overset{i}{o} Q = R$.

ii) If i is bounded difference t-norm

then $i(a,b) = \max(a+b-1, 0)$.

$$= \begin{cases} a+b-1 & , \text{ if } a+b-1 \geq 0 \\ 0 & , \text{ if } a+b-1 \leq 0 \end{cases}$$

$$= \begin{cases} a+b-1 & , \text{ if } a+b \geq 1 \\ 0 & , \text{ if } a+b \leq 1 \end{cases}$$

corresponding to this t-norm we obtain ω_i operator

$\omega_i(a,b) = \sup\{x \mid i(a,b) \leq b\}$

$= \sup\{x \mid a+x-1 \leq b\}$, if $a+x \geq 1$

$= \sup\{x \mid 0 \leq b\}$, if $a+x \leq 1$

$$= \begin{cases} b-a+1 & , a+x \geq 1 \\ 1 & , a+x \leq 1 \quad (\text{i.e, } x \leq 1-a) \end{cases}$$

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$$\omega_i(a,b) = \begin{cases} b-a+1, & b \le a \\ 1, & b \ge a \end{cases} \quad \begin{pmatrix} b-a+1 \text{ if} \\ \text{i.e, } b \le a \end{pmatrix}$$

$$\hat{P}^{-1} = Q \overset{\omega_i}{\circ} R^{-1}$$

$$= \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix} 0.12 & 0.18 & 0.27 \end{bmatrix}$$

$$= \begin{bmatrix} \omega_i(0.1, 0.12) & \omega_i(0.1, 0.18) & \omega_i(0.1, 0.27) \\ \omega_i(0.2, 0.12) & \omega_i(0.2, 0.18) & \omega_i(0.2, 0.27) \\ \omega_i(0.3, 0.12) & \omega_i(0.3, 0.18) & \omega_i(0.3, 0.27) \end{bmatrix}$$

$$\hat{P}^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0.92 & 0.98 & 1 \\ 0.82 & 0.88 & 0.97 \end{bmatrix}$$

hence the greatest solution is,

$$\hat{P} = \begin{bmatrix} 1 & 0.92 & 0.82 \\ 1 & 0.98 & 0.88 \\ 1 & 1 & 0.97 \end{bmatrix}$$

we verify that, $\hat{P} \in S(Q, R)$

consider,

$$\hat{P} \overset{i}{\circ} Q = \begin{bmatrix} 1 & 0.92 & 0.82 \\ 1 & 0.98 & 0.88 \\ 1 & 1 & 0.97 \end{bmatrix} \overset{i}{\circ} \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \end{bmatrix}$$

max(a + b - 1, 0)

= max {0.1, 0.1 ...

= max {0.1, 0.18 ...

max {0.1, 0.2 ...

$$\hat{P} \overset{i}{\circ} Q = \begin{bmatrix} 0.12 \\ 0.18 \\ 0.27 \end{bmatrix}$$

$\hat{P} \overset{i}{\circ} Q = R$, thus $\hat{P} \in S(Q, R)$ i.e, $S(Q,R) \neq \phi$.

$\therefore \hat{P}$ is the greatest sol^n of $P \overset{i}{\circ} Q = R$.

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Que. Obtain the greatest solution of the following fuzzy relation equation, where i = min, product, bounded difference

① $P \overset{i}{\circ} \begin{bmatrix} 0.2 & 0.9 & 0.5 \\ 0 & 0.3 & 0.9 \\ 0.8 & 0 & 0.2 \\ 1 & 0.7 & 0 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.4 & 0.8 \\ 0.3 & 0.4 & 0.9 \\ 1 & 0.7 & 0.2 \end{bmatrix}$

② $P \overset{i}{\circ} \begin{bmatrix} 0.2 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = [0.6 \quad 0.7]$

* Theorem :-

statement : let $P \overset{i}{\circ} Q = R$, be a fuzzy relation eqn & let $P_1, P_2 \in S(Q, R)$
then ① $P_1 \subseteq P \subseteq P_2 \Rightarrow P \in S(Q, R)$
② $P_1 \cup P_2 \in S(Q, R)$

proof :-

① If $P_1, P_2 \in S(Q, R)$ then
$P_1 \overset{i}{\circ} Q = R$ & $P_2 \overset{i}{\circ} Q = R$
If P is any fuzzy relation such that,
$P_1 \leq P \leq P_2$,
then we have,
$P_1 \overset{i}{\circ} Q \subseteq P \overset{i}{\circ} Q \subseteq P_2 \overset{i}{\circ} Q$
$R \subseteq P \overset{i}{\circ} Q \subseteq R$
$\Rightarrow P \overset{i}{\circ} Q = R$
$\Rightarrow P \in S(Q, R)$

② Consider,
$(P_1 \cup P_2) \overset{i}{\circ} Q = (P_1 \overset{i}{\circ} Q) \cup (P_2 \overset{i}{\circ} Q) = R \cup R$
$(P_1 \cup P_2) \overset{i}{\circ} Q = R \cup R$
$\Rightarrow P_1 \cup P_2 \in S(Q, R)$

§ Fuzzy relation eqⁿ based on infimum ω_i composition :-

If $P(X,Y)$, $Q(Y,Z)$ are fuzzy relation on $[0,1]$ & ω_i is a binary operation w.r.t. t-norm i then

$$(P \overset{\omega_i}{\circ} Q)(x,z) = \inf_{y \in Y} \omega_i (P(x,y), Q(y,z))$$

~~if $Q(x,y)$~~

If $Q(y,z)$ & $R(x,z)$ are fuzzy relation & ω_i is a ~~binary operation~~ to determine $P(X,Y)$ s.t.

$$P \overset{\omega_i}{\circ} Q = R.$$

Also, we denote the solⁿ set of these eqⁿs, by,

$$S(Q,R) = \{ P / P \overset{\omega_i}{\circ} Q = R \}$$

* Theorem :-

If $S(Q,R) \neq \phi$ then $\hat{P} = R \overset{\omega_i}{\circ} Q^{-1}$ is the greatest member of $S(Q,R)$

proof

let $P \overset{\omega_i}{\circ} Q = R$

be the relⁿ eqⁿ

by property of inf - ω_i composition.

we have,

$$P \subseteq (P \overset{\omega_i}{\circ} Q) \overset{\omega_i}{\circ} Q^{-1}$$

$$P \subseteq R \overset{\omega_i}{\circ} Q^{-1}$$

$$P \subseteq \hat{P} \quad , \text{ if } P \in S(Q,R)$$

i.e, $\hat{P}$ is greater than every solⁿ of $P \overset{\omega_i}{\circ} Q = R$.

Next we show that, $\hat{P} \in S(Q,R)$

by property of inf. $\overset{\omega_i}{o}$ composition,

$R \subseteq (R \overset{\omega_i}{o} Q^{-1}) \overset{\omega_i}{o} (Q^{-1})^{-1}$

$R \subseteq \hat{P} \overset{\omega_i}{o} Q$ ——— ⓐ

Next, if $P \in S(Q, R)$ then $P \subseteq \hat{P}$

$\Rightarrow P \overset{\omega_i}{o} Q \subseteq \hat{P} \overset{\omega_i}{o} Q$

$\Rightarrow R \supseteq \hat{P} \overset{\omega_i}{o} Q$ $\{Q_1 \subseteq Q_2 \Rightarrow Q_1 \overset{\omega_i}{o} R \supseteq Q_2 \overset{\omega_i}{o} R\}$

i.e. $\hat{P} \overset{\omega_i}{o} Q \subseteq R$ ——— ⓑ

Thus from eqn ⓐ & ⓑ

$R \subseteq \hat{P} \overset{\omega_i}{o} Q \subseteq R$

$\Rightarrow \hat{P} \overset{\omega_i}{o} Q = R$

$\Rightarrow \hat{P} \in S(Q, R)$

~~Hence~~

$\therefore \hat{P} = R \overset{\omega_i}{o} Q^{-1}$ is the greatest member of $S(Q, R)$

Hence the proof.

* Theorem :-

If $S(P, R) \neq \phi$ then $\check{Q} = P^{-1} \overset{i}{o} R$ is the smallest member of $S(P, R)$

proof

let $Q \in S(P, R)$

then $P \overset{\omega_i}{o} Q = R$ & $S(P, R) \neq \phi$

let $\check{Q} = P^{-1} \overset{i}{o} R$

$= P^{-1} \overset{i}{o} (P \overset{\omega_i}{o} Q)$

$\check{Q} \subseteq Q$

Thus $\forall Q \in S(P, R)$, we have

$\check{Q} \subseteq Q$

Now, we show that,

$\check{Q} \in S(P, R)$

Now $\check{Q} = P^{-1} \overset{i}{\circ} R \subseteq Q$

$\Rightarrow \quad P^{-1} \overset{i}{\circ} R \subseteq Q$

$\Rightarrow \quad P \overset{\omega i}{\circ} (P^{-1} \overset{i}{\circ} R) \subseteq P \overset{\omega i}{\circ} Q$

$\Rightarrow \quad P \overset{\omega i}{\circ} \check{Q} \subseteq R$ ——— ⓐ

Next,

$R \subseteq P \overset{\omega i}{\circ} (P^{-1} \overset{i}{\circ} R)$ ——— ⓑ

$\Rightarrow R \subseteq P \overset{\omega i}{\circ} \check{Q}$

then from eq^n ⓐ & ⓑ

$R \subseteq P \overset{\omega i}{\circ} \check{Q} \subseteq R$

i.e $P \overset{\omega i}{\circ} \check{Q} = R$

This shows that,

$\check{Q} \in S(P, R)$

Hence the proof.

* Theorem :-

① $P_1, P_2 \in S(Q, R)$

ⓐ $P_1 \cup P_2 \in S(Q, R)$

ⓑ $P_1 \subseteq P \subseteq P_2 \Rightarrow P \in S(Q, R)$

② If $Q_1, Q_2 \in S(P, R)$ then

ⓐ $Q_1 \cap Q_2 \in S(P, R)$

ⓑ $Q_1 \subseteq Q \subseteq Q_2 \Rightarrow Q \in S(P, R)$

where the relation eq^n R w.r.t. inf-ωi

proof

① If $P_1, P_2 \in S(Q, R)$ then

$P_1 \overset{\omega i}{\circ} Q = R$ &

$P_2 \overset{\omega i}{\circ} Q = R$

ⓐ consider

$(P_1 \cup P_2) \overset{\omega i}{\circ} Q = (P_1 \overset{\omega i}{\circ} Q) \cap (P_2 \overset{\omega i}{\circ} Q)$ $\quad \left\{ \because (\bigcup_j P_j) \overset{\omega i}{\circ} Q \Rightarrow \bigcap_j (P_j \overset{\omega i}{\circ} Q) \right.$

$= R \cap R$

$= R$

$\Rightarrow P_1 \cup P_2 \in S(Q, R)$

(b) If P is any fuzzy rel^n s.t.

$P_1 \subseteq P \subseteq P_2$

$\Rightarrow P_1 \overset{\omega_i}{o} Q \subseteq P \overset{\omega_i}{o} Q \subseteq P_2 \overset{\omega_i}{o} Q$

$\Rightarrow R \subseteq P \overset{\omega_i}{o} Q \subseteq R$

$\Rightarrow P \overset{\omega_i}{o} Q = R$

$\Rightarrow P \in S(Q, R)$

(2) let $Q_1, Q_2 \in S(P, R)$

$\therefore P \overset{\omega_i}{o} Q_1 = R$, $P \overset{\omega_i}{o} Q_2 = R$

(a) consider,

$P \overset{\omega_i}{o} (Q_1 \cap Q_2) = (P \overset{\omega_i}{o} Q_1) \cap (P \overset{\omega_i}{o} Q_2)$

$P \overset{\omega_i}{o} (Q_1 \cap Q_2) = R \cap R$

$P \overset{\omega_i}{o} (Q_1 \cap Q_2) = R$

$\Rightarrow Q_1 \cap Q_2 \in S(P, R)$

(b) If Q is any fuzzy relation s.t.

$Q_1 \subseteq Q \subseteq Q_2$

$\Rightarrow P \overset{\omega_i}{o} Q_1 \subseteq P \overset{\omega_i}{o} Q \subseteq P \overset{\omega_i}{o} Q_2$

$\Rightarrow R \subseteq P \overset{\omega_i}{o} Q \subseteq R$

$\Rightarrow P \overset{\omega_i}{o} Q = R$

$\Rightarrow Q \in S(P, R)$

Hence the proof.

5. Obtain the maximal sol^n of fuzzy rel^n eq^n

$$P \overset{\omega_i}{o} \begin{bmatrix} 0.1 & 0.6 \\ 0.8 & 0.9 \end{bmatrix} = \begin{bmatrix} 0.2 & 1 \\ 0.25 & 1 \end{bmatrix}$$

where i is a t norm given by, (i) $i = \min$ (ii) $i =$ product

Solution :-

(i) $P \overset{\omega_i}{o} Q = R$

$$\therefore Q = \begin{bmatrix} 0.1 & 0.6 \\ 0.8 & 0.9 \end{bmatrix}, \quad R = \begin{bmatrix} 0.2 & 1 \\ 0.25 & 1 \end{bmatrix}$$

$i = \min$

$i(a, b) = \min(a, b)$

$\omega i(a,b) = \{x \mid \min(a,x) \le b\}$

$\omega i(a,b) = \begin{cases} 1 & a \le b \\ b & a > b \end{cases}$

$\hat{P} = R \overset{\omega i}{\circ} Q^{-1}$

$\hat{P} = \begin{bmatrix} 0.2 & 1 \\ 0.25 & 1 \end{bmatrix} \overset{\omega i}{\circ} \begin{bmatrix} \cancel{0.2} & \cancel{0.25} \\ 1 & \cancel{1} \end{bmatrix} \begin{bmatrix} 0.1 & 0.8 \\ 0.6 & 0.9 \end{bmatrix}$

$\hat{P} = \begin{bmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{bmatrix}$ inf(ωi(0.2, 0.1), ωi(1, 0.6)) inf(ωi(0.2, 0.8), ωi(1, 0.9)) inf(ωi(0.25, 0.1), ωi(1, 0.6)) inf(ωi(0.25, 0.8), ωi(1, 0.9))

~~i= product~~ $\hat{P} \in S(Q, R)$

consider,

$\hat{P} \overset{\omega i}{\circ} Q = \begin{bmatrix} \cancel{0.1} & \cancel{0.6} \\ 0.8 & 0.9 \end{bmatrix} \cancel{\overset{\omega i}{\circ}}$

$= \begin{bmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{bmatrix} \overset{\omega i}{\circ} \begin{bmatrix} 0.1 & 0.6 \\ 0.8 & 0.9 \end{bmatrix}$

$= \begin{bmatrix} 0.8 & 1 \\ 0.8 & 1 \end{bmatrix}$ inf(ωi(0.1, 0.1), ωi(0.9, 0.8)) inf(ωi(0.1, 0.6), ωi(0.9, 0.9)) inf(ωi(

$\hat{P} \overset{\omega i}{\circ} Q \ne R$

~~$\hat{P} \overset{\omega i}{\circ} R \notin R$~~

hence. $\hat{P} \notin S(Q, R)$ i.e., $S(Q, R) = \phi$

(ii) i= product

$\omega i(a,b) = \begin{cases} 1 , & a \le b \\ \frac{b}{a} , & a > b \end{cases}$

$\hat{P} = R \overset{\omega i}{\circ} Q^{-1}$

$= \begin{bmatrix} 0.2 & 1 \\ 0.25 & 1 \end{bmatrix} \overset{\omega i}{\circ} \begin{bmatrix} 0.1 & 0.8 \\ 0.6 & 0.9 \end{bmatrix}$

$= \begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 0.9 \end{bmatrix}$

$$\hat{P} = \begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 0.9 \end{bmatrix}$$

we verify that, $\hat{P} \in S(Q, R)$

consider,

$$\hat{P} \overset{\omega_i}{\circ} Q = \begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 0.9 \end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix} 0.1 & 0.6 \\ 0.8 & 0.9 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & 1 \\ 0.29 & 1 \end{bmatrix}$$

$$= R$$

$$\hat{P} \overset{\omega_i}{\circ} Q = R$$

$\hat{P} \in S(Q, R)$ i.e, $S(Q, R) \neq \phi$

It is a ~~fuzzy~~ maximal soln of fuzzy relation.

obtain the minimal soln of fuzzy reln eqn $P \overset{\omega_i}{\circ} Q = R$, where i = product &

$$P = \begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 0.9 \end{bmatrix}, \quad R = \begin{bmatrix} 0.2 & 1 \\ 0.25 & 1 \end{bmatrix}$$

Given eqn, $P \overset{\omega_i}{\circ} Q = R$

To find $\check{Q} \in S(P, R)$

By theorem,

$$\check{Q} = P^{-1} \dot{\circ} R$$

is the minimal soln of $P \overset{\omega_i}{\circ} Q = R$

$$\check{Q} = P^{-1} \dot{\circ} R$$

$$= \begin{bmatrix} 0.5 & 0.4 \\ 0.9 & 0.9 \end{bmatrix} \dot{\circ} \begin{bmatrix} 0.2 & 1 \\ 0.25 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1 & 0.5 \\ 0.225 & 0.9 \end{bmatrix}$$

$$\omega_i(a,b) = \begin{cases} \\ \frac{b}{a} & a > b \end{cases}$$

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Next we show that,

$\check{Q} \in S(P, R)$

$$P \overset{\omega_i}{\circ} \check{Q} = \begin{bmatrix} 0.5 & 0.9 \\ 0.4 & 0.9 \end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix} 0.1 & 0.5 \\ 0.225 & 0.9 \end{bmatrix}$$

$$\omega_i(a,b) = \begin{cases} 1 & a \le b \\ \frac{b}{a} & a > b \end{cases}$$

$$\omega_i(a,b) = \begin{cases} \inf\{\omega_i(0.5, 0.1), \omega_i(0.9, 0.225)\} \;/\; \inf\{\omega_i(0.5, 0.5), \omega_i(0.9, 0.9)\} \\ \inf\{\frac{0.2}{0.2}, 0.25\} \;/\; \inf(1, 1) = 1 \\ \inf\{\omega_i(0.4, 0.1), \omega_i(0.9, 0.225)\} \;/\; \inf(\omega_i(0.4, 0.5), \omega_i(0.9, 0.9)) \\ 1, 1 \qquad\qquad 1 \end{cases}$$

$$\omega_i(a,b) = \begin{bmatrix} 0.2 & 1 \\ 1 & 1 \end{bmatrix}$$

$= R$

$\check{Q} \in S(P, R)$, $S(P, R) \neq \phi$

$\check{Q}$ is minimal solⁿ

Q let i = min, product, bounded diff t-norm determine the maximal or minimal soln if exist for the following

1) $P \overset{\omega_i}{\circ} \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$ — maximal

find $\hat{P} = R \overset{\omega_i}{\circ} Q^{-1}$

(2) $\begin{bmatrix} 1 & 0.9 \\ 0.2 & 0.6 \end{bmatrix} \overset{\omega_i}{\circ} Q = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$ — minimal

~~find~~ find $\check{Q} = P^{-1} \overset{i}{\circ} R$

② (ii) $\begin{bmatrix} 1 & 0.9 \\ 0.2 & 0.6 \end{bmatrix} \overset{\omega_i}{\circ} Q = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$

Find $\check{Q} = \bar{P}^{1} \overset{\cdot}{\circ} R$.

is the minimal soln of $P \overset{\omega_i}{\circ} Q = R$

$\check{Q} = \bar{P}^{1} \overset{\cdot}{\circ} R$

$= \begin{bmatrix} 1 & 0.2 \\ 0.9 & 0.6 \end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}$

Next we show that,

$\check{Q} \in S(P,R)$

$P \overset{\omega_i}{\circ} \check{Q} = \begin{bmatrix} 1 & 0.9 \\ 0.2 & 0.6 \end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}$

$\omega_i(a,b) = \begin{cases} 1 & a \le b \\ b/a & a > b \end{cases}$

$\omega_i(a,b) = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$

inf $(\omega_i(1, 0.2), \omega_i(0.9, 0.6))$
min$(0.2 \quad 0.6)$
$\Rightarrow 0.2$
inf $(\omega_i(0.2, 0.2), \omega_i(0.6, 0.6))$
1 1

$= R$

$\check{Q} \in S(P,R)$, $S(P,R) \neq \phi$

$\check{Q}$ is minimal soln.

(iii) Given, $\cancel{\begin{bmatrix} 1 & 0.9 \\ 0.2 & 0.6 \end{bmatrix} \overset{\omega_i}{\circ} Q = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}}$

(i) Given, $P \overset{\omega_i}{\circ} \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$

$P \overset{\omega_i}{\circ} Q = R$

$Q = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}$, $R = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}$

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$i = \min$

$i(a,b) = \min(a,b)$

$\hat{P} = R \overset{\omega i}{\circ} Q^{-1}$

$\hat{P} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix} \overset{\omega i}{\circ} \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix}$

$\hat{P} =$

$\omega i(a,b) = \begin{cases} 1 & a \le b \\ b & a > b \end{cases}$

$\hat{P} = \begin{bmatrix} \cancel{1} \\ \cancel{0.2} \end{bmatrix} \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$

$\hat{P} \in S(Q,R)$

consider,

$\hat{P} \overset{\omega i}{\circ} Q = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} \overset{\omega i}{\circ} \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}$

$= \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$

$\hat{P} \overset{\omega i}{\circ} Q \neq R$

(ii) $i =$ product

$\omega i(a,b) = \begin{cases} 1 & , \quad a \le b \\ b/a & , \quad a > b \end{cases}$

$\hat{P} \overset{\omega i}{\circ}$

$\hat{P} = R \overset{\omega i}{\circ} Q^{-1}$

$= \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} \overset{\omega i}{\circ} \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix}$

$\hat{P} = \begin{bmatrix} 1 \\ 0.2 \end{bmatrix}$

we verify that, $\hat{P} \in S(Q,R)$

consider,

$$\hat{P} \overset{\omega_i}{\circ} Q = \begin{bmatrix}1\\0.2\end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix}0.2\\0.6\end{bmatrix}$$

$$= \begin{bmatrix}0.2\\1\end{bmatrix}$$

$$= R$$

$$\hat{P} \overset{\omega_i}{\circ} Q = R$$

$$\hat{P} \in S(Q,R)$$

$$S(Q,R) \neq \phi$$

(iii) i = bounded diff t-norm

$$i(a,b) = \max(a+b-1, 0)$$

$$= \begin{cases} a+b-1, & \text{if } a+b-1 \geq 0 \\ 0, & \text{if } a+b-1 \leq 0 \end{cases}$$

$$= \begin{cases} a+b-1, & \text{if } a+b \geq 1 \\ 0, & \text{if } a+b \leq 1 \end{cases}$$

$$\omega_i(a,b) = \begin{cases} b-a+1, & b \leq a \\ 1, & b \geq a \end{cases}$$

$$\hat{P}^{-1} = \cancel{Q \overset{\omega_i}{\circ} R^{-1}} \quad R \overset{\omega_i}{\circ} Q^{-1}$$

$$= \begin{bmatrix}0.2\\1\ \cancel{0.6}\end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix}0.2, & 0.6\end{bmatrix}$$

$$= \begin{bmatrix}\omega_i(0.2, 0.2)\\ \omega_i(\cancel{0.6}\ 1, \cancel{0.2}\ 0.6)\end{bmatrix}$$

$$\hat{P}^{-1} = \begin{bmatrix}\cancel{1.8}\\ \cancel{0.6}\end{bmatrix} = \begin{bmatrix}1\\ \cancel{0.2}\end{bmatrix} \begin{bmatrix}\cancel{0}\end{bmatrix}$$

$$\hat{P} = \begin{bmatrix}\cancel{0.6}\\ \cancel{1}\end{bmatrix} \begin{bmatrix}1\\0.2\end{bmatrix}$$

we verify that, $\hat{P} \in \mathcal{S}(Q, R)$

consider,

$$\hat{P} \dot{\circ} Q = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} \dot{\circ} \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}$$

$$= R$$

* Approximate Solution :-

Def^n:- Consider a fuzzy relation eq^n $P \dot{o} Q = R$.

for some eq^n, if there is no exact sol^n for P in such case we modify Q & R in to Q' & R' s.t. $P \dot{o} Q' = R'$ has a sol^n P' i.e, $P' \dot{o} Q' = R'$.

This sol^n P' is called an approximate sol^n of given eq^n $P \dot{o} Q = R$

* Def^n:-

MCQ defn 2 mark

A fuzzy rel^n $\tilde{P}$ is called an approximate sol^n of $P \dot{o} Q = R$ if the following condition holds

① $\exists$ $Q' \supseteq Q$ & $R' \subseteq R$
s.t. $\tilde{P} \dot{o} Q' = R'$

② if $\exists$ P'', Q'', R'' s.t. $Q'' \in$
$Q \subseteq Q'' \subseteq Q'$ & $R' \subseteq R'' \subseteq R$ & $P'' \dot{o} Q' = R''$
then $Q'' = Q'$ & $R'' = R'$

⊙ Note :-

1) The cond^n ① in the above def^n means that approximate sol^n of $P \dot{o} Q = R$ is obtained by making Q larger & R smaller

2) The cond^n ② means Q' & R' are the closest relation to Q & R for which solution exist.

IMP
6 marks

* Example

Find approximate soln of $P \overset{i}{\circ} Q = R$

$$Q = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}, \; R = [0.5 \;\; 0.6] \;\&\; i = \min$$

Soln:-

let, $\hat{P} = (Q \overset{\omega i}{\circ} R^{-1})^{-1}$

$$\Rightarrow (\hat{P})^{-1} = Q \overset{\omega i}{\circ} R^{-1}$$

$$= \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \overset{\omega i}{\circ} \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}$$

$$\omega i(a,b) = \begin{cases} 1 & a \le b \\ b & a > b \end{cases}$$

$$(\hat{P})^{-1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\inf\{\omega i(0.1, 0.5), \omega i(0.3, 0.6)\}$
$\min(1, 1)$
$\inf\{\omega i(0.2, 0.5), \omega i(0.4, 0.6)\}$
$\min(1, 1)$

$$\hat{P} = [1, 1]$$

Now we have to verify $\hat{P} \in S(Q, R)$

$$\hat{P} \overset{i}{\circ} Q = [1, 1] \overset{i}{\circ} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$$

$\inf\{\omega i(1, 0.1), (1, 0.2)\}$

$$= [0.2 \;\; 0.4]$$

$\inf\{\omega i(1, 0.3), (1, 0.4)\}$

$$\ne R$$

$$\hat{P} \notin S(Q, R)$$

Thus fuzzy reln eqn $P \overset{i}{\circ} Q = R$ does not have solution ~~now to fin~~

Now, to find approximate soI^n we reduce $R = [0.5, 0.6]$ to $R' = [0.2, 0.4]$ $(\because R' \leq R)$

then $\hat{P} = (Q \overset{\omega}{\circ}' R'^{-1})^{-1} = [1, 1]$

$\hat{P} \overset{i}{\circ} Q = R'$

Now assume that, $\exists$ R'' s.t $R' \subset R'' \subset R$

$$P'' \overset{i}{\circ} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} = R''$$ has soI^n P''.

then,

$\max(0.1P_1, 0.2P_2) = r_1$

$\max(0.3P_1, 0.4P_2) = r_2$

where, $P'' = [P_1, P_2]$ &

$R'' = [r_1, r_2]$

this eq^n are satisfied only if ~~$r_1 \leq$~~ $r_1 \leq 0.2$ & $r_2 \leq 0.4$, $R'' \subseteq R'$

$\therefore R' = R''$

Thus, $\hat{P}$ is an approximate soI^n of original fuzzy eq^n.

imp March 2016

Theorem :-

let $P \overset{i}{\circ} Q = R$ be a fuzzy relation eq^n then $\tilde{P} = (Q \overset{\omega}{\circ}' R^{-1})^{-1}$ is the greatest approximate soI^n of $P \overset{i}{\circ} Q = R$ □

more

proof -

let $P \overset{i}{\circ} Q = R$ be a fuzzy reI^n eq^n

let $\tilde{P} = (Q \overset{\omega}{\circ}' R^{-1})^{-1}$ be a fuzzy reI^n

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Take $Q' = Q$ & $R' = (Q \overset{\omega i}{\circ} R^{-1})^{-1} \dot{\circ} Q'$

then, $\tilde{P} \dot{\circ} Q' = (Q \overset{\omega i}{\circ} R^{-1})^{-1} \dot{\circ} Q' = R'$

i.e, $\tilde{P} \dot{\circ} Q' = R'$

$\Rightarrow \tilde{P} \in S(Q', R')$

we show that, $R' \subseteq R$

consider, $R' = (Q \overset{\omega i}{\circ} R^{-1})^{-1} \dot{\circ} Q$

$= [Q^{-1} \dot{\circ} (Q \overset{\omega i}{\circ} R^{-1})]^{-1}$

$\subseteq (R^{-1})^{-1}$

$R' \subseteq R$

Next,

Assume that, $\exists$ P'' & R'' s.t.

$R' \subseteq R'' \subseteq R$ & $P'' \dot{\circ} Q = R''$

then, $(Q \overset{\omega i}{\circ} R''^{-1})^{-1} \subseteq S(Q, R'')$

since,

$(Q \overset{\omega i}{\circ} R''^{-1})^{-1}$ is the greatest soln of

$P'' \dot{\circ} Q = R''$

we have, $(Q \overset{\omega i}{\circ} R''^{-1})^{-1} \dot{\circ} Q = R''$

Now, $R' \subseteq R'' \subseteq R$

$\Rightarrow R'^{-1} \subseteq R''^{-1} \subseteq R^{-1}$

$\Rightarrow Q \overset{\omega i}{\circ} R^{-1} \subseteq Q \overset{\omega i}{\circ} R''^{-1} \subseteq Q \overset{\omega i}{\circ} R^{-1}$

but, $Q \overset{\omega i}{\circ} R'^{-1} = Q \overset{\omega i}{\circ} [(Q \overset{\omega i}{\circ} R^{-1})^{-1} \dot{\circ} Q']^{-1}$

$= Q \overset{\omega i}{\circ} [Q^{-1} \dot{\circ} (Q \overset{\omega i}{\circ} R^{-1})]$

$Q \overset{\omega i}{\circ} R'^{-1} \supseteq Q \overset{\omega i}{\circ} R^{-1}$ $\{\because R \subseteq P \dot{\circ} (P^{-1} \overset{\omega i}{\circ} R)$

i.e, $Q \overset{\omega i}{\circ} R^{-1} \subseteq Q \overset{\omega i}{\circ} R'^{-1}$

$\therefore$ applying this result to, $R' \subseteq R'' \subseteq R$

we have,

$Q \overset{\omega i}{\circ} R^{-1} \subseteq Q \overset{\omega i}{\circ} R'^{-1} \subseteq Q \overset{\omega i}{\circ} R''^{-1} \subseteq Q \overset{\omega i}{\circ} R^{-1}$

$\Rightarrow Q \overset{\omega i}{\circ} R''^{-1} = Q \overset{\omega i}{\circ} R^{-1}$

$\therefore \quad R'' = (Q \overset{\omega i}{o} R''^{-1})^{-1} \dot{o}\, Q$

$\qquad = (Q \overset{\omega i}{o} R'^{-1})^{-1} \dot{o}\, Q = R'$

$R'' = R'$

Thus,

i) $\exists\ Q' = Q$ & $R' \subseteq R$ s.t. $\tilde{P} \dot{o} Q' = R'$

ii) $\exists\ P'', Q''$ & R'' s.t. $Q \subseteq Q'' \subseteq Q'$ & $R' \subseteq R'' \subseteq R$

with $P'' \dot{o} Q'' = R''$ then $R'' = R'$ & $Q'' = Q'$

hence $\tilde{P}$ is an approximate solution of $P \dot{o} Q = R$.

Finally we show that,

$$\tilde{P} = (Q \overset{\omega i}{o} R^{-1})^{-1},$$

is the greatest approximate sol^n

for this assume that,

$\tilde{P}'$ is another approximate sol^n of $P \dot{o} Q = R$

then $\exists\ Q \subseteq Q'$, $R' \subseteq R$ & $\tilde{P}' \dot{o} Q' = R'$

i.e. $\tilde{P}' \in S(Q', R')$

but, $(Q' \overset{\omega i}{o} R'^{-1})^{-1}$ is the greatest element of $S(Q', R')$

$\therefore \quad \tilde{P}' \subseteq (Q' \overset{\omega i}{o} R'^{-1})^{-1}$

now,

$Q \subseteq Q'$, & $R' \subseteq R$

hence $(Q' \overset{\omega i}{o} R'^{-1})^{-1} \subseteq (Q \overset{\omega i}{o} R^{-1})^{-1} = \tilde{P}$

$\Rightarrow \quad \tilde{P}' \subseteq (Q' \overset{\omega i}{o} R'^{-1})^{-1} \subseteq \tilde{P}$

$\Rightarrow \quad \tilde{P}' \subseteq \tilde{P}$

This shows that, $\tilde{P}$ is the greatest approximate sol^n of $P \dot{o} Q = R$

Hence the proof.

⊙ Remark

A relation eq^n $P \overset{i}{\circ} Q = R$
if $\hat{P} = (Q \overset{\omega i}{\circ} R^{-1})^{-1}$ is a sol^n then it is the greatest sol^n of eq^n.
Further if $\hat{P}$ is not a solution then it is the greatest approximate sol^n

* Def^n

Equality Index (Goodness index)
let $\|P\|$ denotes the degree of truth of the fuzzy praposition P then for any given fuzzy sets defined on the ~~same~~ universal set X. we have $\|A = B\|$ called as equality index or a resonable measure of the degree to which the two sets are equal.
we have,
$\|A = B\| = \min\{\|A \subseteq B\|, \|A \supseteq B\|\}$
here,
$\|A \subseteq B\|$ satisfying the following cond^n
1) $\|A \subseteq B\| \in [0,1]$
2) $\|A \subseteq B\| = 1$ if $A(x) \leq B(x)$ $\forall x \in X$ (i.e., $A \subseteq B$)
3) If $C \subseteq B$ then
$\|A \subseteq C\| \leq \|A \subseteq B\|$
$\|C \subseteq A\| \geq \|B \subseteq A\|$

This conditions are neecessory for acceptable def^n of $\|A \subseteq B\|$ but they are not sufficient to determine them uniquely.

There are several def^n of $\|A \subseteq B\|$ one of the def^n is given by,

$$\|A \subseteq B\| = \frac{|A \cap B|}{|A|}$$

where, $\|A \cap B\|$ denotes scalar cardinality of sigma count & intersection denotes standard fuzzy intersection.

Based on ω_i-composition we have,

$\|A \subseteq B\| = \inf_x \omega_i (A(x), B(x))$ if X is not finite

These two def^n satisfies the cond^n ①, ②, ③

Hence the equality index is defined by, (w.r.t. ω_i-composition)

$$\|A = B\| = \min(\|A \subseteq B\|, \|B \subseteq A\|)$$
$$= \inf_x \min(\omega_i(A(x), B(x)), \omega_i(B(x), A(x)))$$

* def^n:-

If P' is an approximat sol^n, $P \mathbin{\dot{\circ}} Q = R$ then goodness index of P' is defined by

$G(P') = \| P' \mathbin{\dot{\circ}} Q = R \|$

The greater the equality degree & $G(P')$, the better the approximation P' for the eq^n $P \mathbin{\dot{\circ}} Q = R$

$G(P') = 1$ iff P' is the exact sol^n of the eq^n $P \mathbin{\dot{\circ}} Q = R$.

* Theorem :-

The fuzzy relⁿ → The fuzzy rel^n $\tilde{P} = (Q \overset{\omega}{\circ} R^{-1})^{-1}$ is the best approximation in terms of goodness index G of the fuzzy rel^n $P \circ Q = R$.

IMP

proof :-

To prove this th^m. we show that if P' is any other approximate sol^n of $P \dot{\circ} Q = R$ then $G(P') \leq G(\tilde{P})$

Now by def^n of goodness index, we have,

$$G(\tilde{P}) = \| \tilde{P} \dot{\circ} Q = R \|$$

$$\Rightarrow G(\tilde{P}) = \min(\| \tilde{P} \dot{\circ} Q \subseteq R \|, \| \tilde{P} \dot{\circ} Q \supseteq R \|)$$

but we have,

$$\tilde{P} = (Q \overset{\omega}{\circ} R^{-1})^{-1}$$

then, $\tilde{P} \dot{\circ} Q = (Q \overset{\omega}{\circ} R^{-1})^{-1} \dot{\circ} Q$

$$= [Q^{-1} \dot{\circ} (Q \overset{\omega}{\circ} R^{-1})]^{-1}$$

$$\subseteq (R^{-1})^{-1}$$

$$\tilde{P} \dot{\circ} Q \subseteq R$$

Hence, $\| \tilde{P} \dot{\circ} Q \subseteq R \| = 1$

$$\therefore G(\tilde{P}) = \min(\| \tilde{P} \dot{\circ} Q \subseteq R \|, \| \tilde{P} \dot{\circ} Q \supseteq R \|)$$

$$= \min(1, \| \tilde{P} \dot{\circ} Q \supseteq R \|)$$

$$= \| \tilde{P} \dot{\circ} Q \supseteq R \|$$

If P' is any other approximate sol^n of $P \dot{\circ} Q = R$ then $P' \subseteq \tilde{P} = (Q \overset{\omega}{\circ} R^{-1})^{-1}$

$$\Rightarrow P' \dot{\circ} Q \subseteq \tilde{P} \dot{\circ} Q$$

$$\Rightarrow P' \dot{\circ} Q \subseteq R$$

$$\| \tilde{P} \dot{\circ} Q \supseteq R \| \geq \| P' \dot{\circ} Q \supseteq R \|$$

consider,

$$G(\tilde{P}) = \| \tilde{P} \dot{o} Q \supseteq R \|$$
$$\geq \| P' \dot{o} Q \supseteq R \|$$
$$\geq \min \{ \| P' \dot{o} Q \subseteq R \|, \| P' \dot{o} Q \supseteq R \| \}$$
$$= G(P')$$

$\Rightarrow G(\tilde{P}) \geq G(P')$

i.e, $G(P') \leq G(\tilde{P})$

this shows that, $\tilde{P}$ is the best approximation of the solution of $P \dot{o} Q = R$ w.r.t. the goodness index G.

Hence the proof.

* Solvability Index

Solvability index is characterizes the ease in solving fuzzy relation of type $P \dot{o} Q = R$

The solvability index δ is defined by,

$$\delta = \sup_{P} \{ \| P \dot{o} Q = R \| \}$$

Note:-

MCQ 1) If $\delta = 1$ then the equation $P \dot{o} Q = R$ is solvable for P

2) If $\delta < 1$ then the equation $P \dot{o} Q = R$ has no exact solution

3) Smaller the value of δ the more difficult to solve the equation approximately.

4) If $\tilde{P}$ is the greatest approximate sol^n then $G(\tilde{P}) < \delta$

i.e, $G(\overset{\approx}{D})$ is the lower bound for δ.

Theorem :-

let δ be the solvability index of $P \dot{o} Q = R$ defined by

$$\delta = \sup_{P \in \overset{\approx}{F}(X \times Y)} \{ \| P \dot{o} Q = R \| \}$$

Then,

$$\delta \leq \inf_{z \in Z} \omega_i \left(\sup_{x \in X} R(x,z), \sup_{y \in Y} Q(y,z) \right)$$

proof :-

let $P \in \overset{\approx}{F}(X \times Y)$

then, $\| P \dot{o} Q = R \| = \min \{ \| P \dot{o} Q \subseteq R \|, \| P \dot{o} Q \supseteq R \| \}$

$$\Rightarrow \| P \dot{o} Q = R \| \leq \| P \dot{o} Q \supseteq R \|$$

$$= \inf_{(x,z) \in X \times Z} \omega_i \left(R(x,z), P \dot{o} Q(x,z) \right)$$

$\{ \| A \subseteq B \| = \inf_{x} \omega_i (A(x), \ldots$

$$= \inf_{x} \inf_{z} \omega_i \left(R(x,z), P \dot{o} Q(x,z) \right)$$

$$= \inf_{z} \inf_{x} \omega_i \left(R(x,z), P \dot{o} Q(x,z) \right)$$

$$= \inf_{z} \omega_i \left(\sup_{x} R(x,z), \sup_{x} (P \dot{o} Q)(x,z) \right)$$

$$\leq \inf_{z} \inf_{x} \omega_i \left(R(x,z), P \dot{o} Q(x,z) \right)$$

where $1(x,y) = 1 \quad \forall (x,y)$

[since $P(x,y) \leq 1(x,y) \Rightarrow P \dot{o} Q(x,z) \leq (1 \dot{o} Q)(x,z)$

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$$= \inf_{z} \omega_i \left(\sup_{x} R(x,z), \sup_{y} i\,(1(x,y), Q(y,z)) \right)$$

{ by defn of sup i comp

$$= \inf_{z} \omega_i \left(\sup_{x} R(x,z), \sup_{y} Q(y,z) \right)$$

since $i(0,1)=$

$$\| P \overset{i}{\circ} Q = R \| \leq \inf_{z} \omega_i \left(\sup_{x} R(x,z), \sup_{y} Q(y,z) \right)$$

$$\sup_{P \in \tilde{F}(x,y)} \| P \overset{i}{\circ} Q = R \| \leq \inf_{z \in Z} \omega_i \left(\sup_{x} R(x,z), \sup_{y} Q(y,z) \right)$$

$$\boxed{\delta \leq \inf_{z \in Z} \omega_i \left(\sup_{x} R(x,z), \sup_{y} Q(y,z) \right)}$$

Ex. let $i =$ min, pro, bounded diff, Find an approximate soln for each of the following fuzzy reln eqn. Find the goodness index for the approximate soln & also obtain the solvability index for each eqn.

6 marks

① $P \overset{i}{\circ} \begin{bmatrix} 0.2 & 0.3 \\ 0.2 & 0.5 \end{bmatrix} = [0.6 \;\; 0.7]$

② $\begin{bmatrix} 0.6 & 0.4 \\ 0.8 & 0.2 \end{bmatrix} \overset{i}{\circ} Q = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix}$

③ $P \overset{i}{\circ} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} = [0.5 \;\; 0.6]$

5) Solution: (3)

Given in fuzzy rel^n eq^n

$$P \overset{i}{\circ} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} = [0.5 \quad 0.6]$$

where, i = product t-norm

We know, the sol^n of eq^n does not exist if $\max_j q_{jk} < \max_i r_{ik}$, for some k

hence,

$$\max_j q_{j1} = \max\{q_{11}, q_{21}\} = \max\{0.1, 0.2\} = 0.2$$

& $\max_i r_{i1} = \max\{r_{11}\} = 0.5$

Thus, $\max_j q_{j1} < \max_i r_{i1}$

hence the given has no solution,

$\therefore$ The greatest approximate solution of the eq^n is

$$\tilde{P} = (Q \overset{\omega_i}{\circ} R^{-1})^{-1}$$

Now, $R = [0.5 \quad 0.6]$

$$R^{-1} = \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}$$

consider, $Q \overset{\omega_i}{\circ} R^{-1} = \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix} \overset{\omega_i}{\circ} \begin{bmatrix} 0.5 \\ 0.6 \end{bmatrix}$

For i = product

$$\omega_i(a,b) = \begin{cases} 1 & \text{if } a \le b \\ \frac{b}{a} & \text{if } a > b \end{cases}$$

$$Q \overset{\omega i}{\circ} R^{-1} = \begin{bmatrix} \inf\{\omega i(0.1, 0.5), \omega i(0.3, 0.6)\} \\ \inf\{\omega i(0.2, 0.5), \omega i(0.4, 0.6)\} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$(Q \overset{\omega i}{\circ} R^{-1})^{-1} = [1, 1]$$

$$\tilde{P} = [1 \ 1]$$

Thus $\tilde{P} = [1 \ 1]$ is the greatest approximate solution of $P \overset{\omega i}{\circ} Q = R$.

Now,

$$G(\tilde{P}) = \| \tilde{P} \dot{\circ} Q \equiv R \|$$

$$= \| R \subseteq \tilde{P} \dot{\circ} Q \|$$

$$= \inf_{(x,z) \in X \times Z} \omega i (R(x,z), \tilde{P} \dot{\circ} Q(x,z))$$

consider,

product cha max-min

$$\tilde{P} \dot{\circ} Q = [1 \ 1] \dot{\circ} \begin{bmatrix} 0.1 & 0.3 \\ 0.2 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$$

$$G(\tilde{P}) = \inf\{ \omega i(R(x,z), \tilde{P} \dot{\circ} Q(x,z)), \omega i(R(x,z_2), \tilde{P} \dot{\circ} Q(x,z_2))\}$$

$$= \inf\{ \omega i(0.5, 0.2), \omega i(0.6, 0.4)\}$$

$$= \inf\{0.4, 0.66\}$$

$$G(\tilde{P}) = 0.4$$

Also for solvability index,

$$\delta \le \inf_{x \in Z} \omega i \left(\sup_x R(x,z), \sup_y Q(y,z) \right)$$

$$= \inf_{z \in Z} \left\{ \begin{array}{l} \omega_i(R(x, z_1), Q(y, z)), \\ \omega_i(R(x, z_2), Q(y, z_2)) \end{array} \right\}$$

$$= \inf_{\cancel{z \in Z}} \{ \omega_i(0.5, 0.2), \omega_i(0.6, 0.4) \}$$

$$= \inf \{ \cancel{\omega_i(0.5)}, 0.4, 0.66 \}$$

$$= 0.4$$

$$S \leq 0.4$$

But $G(\tilde{P}) \leq \delta \leq 0.4$

$$0.4 \leq \delta \leq 0.4$$

$$\boxed{\delta = 0.4}$$

* Unit - 3 *

* Fuzzy logic

logic is a study of methods & rules of reasoning. classical logic deals with proposition which are either true or false. Two valued logic variables assumes 2^2 prospective truth values & hence there are $2^{2^2} = 2^4 = 16$ logic fn of this variables.

If a & b are two logic formulas then the following rules are as inferrance rule.

① $a \wedge (a \Rightarrow b) \Rightarrow b$ (Modus Ponens)

② $\bar{b} \wedge (a \Rightarrow b) \Rightarrow \bar{a}$ (Modus Tollens)

③ $(a \Rightarrow b) \wedge (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$ (Hypothetical syllogism)

* Fuzzy Proposition :-

classical proposition is either true or false on the other hand fuzzy proposition have degree of truth between zero & one. Following are four types of simple fuzzy propositions

① Unconditional & unqualified proposition

This type of fuzzy proposition is given by P: $\mathcal{V}$ is F

where $\mathcal{V}$ is universel set & F is fuzzy set of $\mathcal{V}$ v and v is variable whose value lies in $\mathcal{V}$.

The degree of truth of this proposition P is given by $T(P) = F(v)$

e.g. P : temp (v) is high

P : 60° is high temp.

$T(P) = F(60°) = 0.8$

2) Unconditional & Qualified proposition :-

This type of proposition is given by
$P : r$ is F is S.
where, S is truth qualifier proposition.
Truth value of this proposition is
$T(P) = S = F(r)$
P = temp(r) is high (F) is true or very true or verly true or not true.
The qualified funⁿ discribe the various qualities of statement.

3) Conditional & Unqualified Proposition
This type of proposition are of the form
P: If X is A then Y is B
where, X & Y are variables whose values are X & Y & A & B are fuzzy sets truth value of this proposition is given by
$T(P) = J(A(x), B(y))$
where, J is fuzzy implication

(1) Ex. If the salary (x) is good (A) then the job(y) is attractive (B).
This proposition can be expressed in the form of fuzzy relation. $R(x,y)$ where
$R : X \rightarrow [0,1]$
The value of R is determined by formula
$R(x,y) = J(A(x), B(y))$
where, J is binary operation on $[0,1]$

$A = \frac{0.1}{x_1} + \frac{0.8}{x_2} + \frac{1}{x_3}$, $B = \frac{0.5}{y_1} + \frac{1}{y_2}$

then for fuzzy implication

$J(a,b) = \min(1, 1-a+b)$

Find the fuzzy rel^n ~~are~~ R.

$\Rightarrow$ We have.

$R(a,b) = J(A(a), B(b))$

$= \min(1, 1 - A(a) + B(b))$

$X \times Y = \{x_1, x_2, x_3\} \times \{y_1, y_2\}$

$= \{(x_1, y_1), (x_1, y_2), (x_2, y_1) (x_2, y_2), (x_3, y_1), (x_3, y_2)\}$

$R(x_1, y_1) = J(A(x_1), B(y_1))$

$= J(0.1, 0.5)$

$= \min(1, 1-0.1+0.5)$

$= \min(1, 1.4)$

$= 1$

$R(x_1, y_2) = J(A(x_1), B(y_2))$

$= J(0.1, 1)$

$= \min(1, 1-0.1+1)$

$= \min(1, 1.9)$

$= 1$

$R(x_2, y_1) = J(0.8, 0.5)$

$= \min(1, 1-0.8+0.5)$

$= \min(1, 0.7)$

$= 0.7$

$R(x_2, y_2) = J(0.8, 1)$

$= \min(1, 1-0.8+1)$

$= \min(1, 1.2)$

$= 1$

$$R(x_3, y_1) = J(1, 0.5)$$
$$= \min(1, 1 - 1 + 0.5)$$
$$= \min(1, 0.5)$$
$$= 0.5$$

$$R(x_3, y_2) = J(1, 1)$$
$$= \min(1, 1 - 1 + 1)$$
$$= \min(1, 1)$$
$$= 1$$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 0.7 & 1 \\ 0.5 & 1 \end{bmatrix} \end{array}$$

(2) $A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$

$B = \frac{1}{y_1} + \frac{0.4}{y_2}$

$J(a, b) = \min(1, 1 - a + b)$

Find the fuzzy relation R.

* Inference from conditional proposition

let A' & B' be the fuzzy sets on X & Y let R be the fuzzy relⁿ on $X \times Y$. If R & A' are given we can obtain B' by the eqⁿ

$B' = A' \circ R$

i.e., $B'(y) = \sup_{x} \min \{ A'(x), R(x) \}$

For conditional & unqualified proposition

p: If X is A then Y is B, we have

$R(x,y) = J(A(x), B(y))$

where, J is fuzzy implication

using fuzzy modus ponens we have,

p: If X is A then Y is B

q: X is A'

conclusion Y is B', where $B' = A' \circ R$ &

$R(x,y) = J(A(x), B(y))$

Ex. let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$

Assume that fuzzy proposition is if X is A then Y is B

where $A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$

$B = \frac{1}{y_1} + \frac{0.4}{y_2}$

If a given fact is expressed by the proposition "X is A'", where

$A' = \frac{0.3}{x_1} + \frac{0.8}{x_2} + \frac{0.4}{x_3}$, using generalized modus ponens.

Derive the conclusion B' (use lucasiewicz implication) given by $J(a,b) = \min(1, 1-a+b)$)

Solution :-

let R be a Fuzzy relation on X×Y.

Defined by,

$R(x,y) = J(A(x), B(y))$

$\therefore R(x_1,y_1) = J(A(x_1), B(y_1))$

$= J(0.5, 1)$

$= \min(1, 1-0.5+1)$

$= 1$

$R(x_1, y_2) = J(0.5, 0.4)$

$= \min(1, 1-0.5+0.4)$

$= 0.9$

$R(x_2, y_1) = J(1, 1)$

$= \min(1, 1-1+1)$

$= 1$

$J(x_2, y_2) = J(1, 0.4)$

$= \min(1, 1-1+0.4)$

$= 0.4$

$J(x_3, y_1) = J(0.6, 1)$

$= \min(1, 1-0.6+1)$

$= 1$

$J(x_3, y_2) = J(0.6, 0.4)$

$= \min(1, 1-0.6+0.4)$

$= 0.8$

$$R(X,Y) = \begin{bmatrix} 1 & 0.9 \\ 1 & 0.4 \\ 1 & 0.8 \end{bmatrix}$$

By Modus ponens.

If X is A' then Y is B'

& X is A' ⇒ Y is B'

where B' = A' o R

min → max → ·

$$B' = \begin{bmatrix} 0.3 & 0.8 & 0.4 \end{bmatrix} \circ \begin{bmatrix} 1 & 0.9 \\ 1 & 0.4 \\ 1 & 0.3 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0.8 & 0.4 \end{bmatrix}$$

i.e., $B' = \frac{0.8}{y_1} + \frac{0.4}{y_2}$

×o×

Ex2) Find the conclusion B' in the above example using $J(a,b) = \omega i(a,b)$ for i = min & i = product compare the conclusion in all the three cases. $A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$, $B = \frac{1}{y_1} + \frac{0.4}{y_2}$

* Generalised Modus Tollens.

Rule:

If X is A then Y is B, Fact Y is B'

conclusion X is A'

wher $A' = R \circ B'$ (max - min comp)

Ex1) If the rule is

If X is A then Y is B

The given fact is Y is B'

where $B' = \frac{0.9}{y_1} + \frac{0.7}{y_2}$

obtain conclusion "X is A'" using generalized modus Tollens (use lucasiewicz implication operation)

$A = \frac{0.5}{x_1} + \frac{1}{x_2} + \frac{0.6}{x_3}$, $B = \frac{1}{y_1} + \frac{0.4}{y_2}$

is true

⇒ If x is A then Y is B then the correspond fuzzy relation ~~are~~ R is given by,

$R(x, y) = J(A(x), B(y))$

Then,

$$R = \begin{bmatrix} 1 & 0.9 \\ 1 & 0.4 \\ 1 & 0.8 \end{bmatrix}$$

given fact is $B' = \frac{0.9}{y_1} + \frac{0.7}{y_2}$

Then $A' = R \circ B'$

$$= \begin{bmatrix} 1 & 0.9 \\ 1 & 0.4 \\ 1 & 0.8 \end{bmatrix} \circ \begin{bmatrix} 0.9 \\ 0.7 \end{bmatrix}$$

max min composition

$$= \begin{bmatrix} 0.9 \\ 0.9 \\ 0.9 \end{bmatrix}$$

i.e., $A' = \frac{0.9}{x_1} + \frac{0.9}{x_2} + \frac{0.9}{x_3}$

② Solve the following problem by using modus ~~Tollens~~ ponens, where, $A = \frac{0.6}{x_1} + \frac{1}{x_2} + \frac{0.9}{x_3}$,

$B = \frac{0.6}{y_1} + \frac{0.7}{y_2}$ & given that

$A' = \frac{0.5}{x_1} + \frac{0.9}{x_2} + \frac{1}{x_3}$, use modus tollens to get the fact A' from conclusion B'.

→ let R be the fuzzy reln on X × Y

$R(x, y) = J(A(x), B(y))$

$R(x_1, y_1) = J(0.6, 0.6)$
$= \min(1, 1 - 0.6 + 0.6)$
$= 1$

$R(x_1, y_2) = J(0.6, 0.7)$
$= \min(1, 1 - 0.6 + 0.7)$
$= 1$

$R(x_2, y_1) = J(1, 0.6)$
$= \min(1, 1 - 1 + 0.6)$
$= 0.6$

$R(x_2, y_2) = J(1, 0.7)$
$= \min(1, 1 - 1 + 0.7)$
$= 0.7$

$R(x_3, y_1) = J(0.9, 0.6)$
$= \min(1, 1 - 0.9 + 0.6)$
$= 0.7$

$R(x_3, y_2) = J(0.9, 0.7)$
$= \min(1, 1 - 0.9 + 0.7)$
$= 0.8$

$$R = \begin{bmatrix} 1 & 1 \\ 0.6 & 0.7 \\ 0.7 & 0.8 \end{bmatrix}$$

by Modus ponens.

$$B' = A' \circ R$$
$$= [0.5 \quad 0.9 \quad 1] \begin{bmatrix} 1 & 1 \\ 0.6 & 0.7 \\ 0.7 & 0.8 \end{bmatrix}$$
$$= [0.7 \quad 0.8]$$

i.e, $B' = \frac{0.7}{y_1} + \frac{0.8}{y_2}$

Then $A' = R \circ B'$

$$= \begin{bmatrix} 1 & 1 \\ 0.6 & 0.7 \\ 0.7 & 0.8 \end{bmatrix} \circ \begin{bmatrix} 0.7 \\ 0.8 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0.8 \\ 0.7 \\ 0.8 \end{bmatrix}$$

$$A' = \frac{0.8}{x_1} + \frac{0.7}{x_2} + \frac{0.8}{x_3}$$

* Fuzzy implications :-

A fuzzy implication J is a function

$J : [0,1] \times [0,1] \longrightarrow [0,1]$

s.t. $J(1,0) = 0$, $J(1,1) = J(0,1) = J(0,0) = 1$

e.g.

The lukaseiwiez implication function given by $J(a,b) = \min(1, 1-a+b)$

Here, $J(1,0) = \min(1, 1-1+0)$

$= 0$

$J(1,1) = \min(1, 1-1+1) = 1$

$J(0,1) = \min(1, 1-0+1) = 1$

$J(0,0) = \min(1, 1-0+0) = 1$

Thus lukaseiwiez operator is a fuzzy implication.

Similarly, we have different types of fuzzy implications given by,

① $J(a,b) = \omega_i(a,b) = \sup\{x \mid i(a,x) \le b\}$
here i is a t-norm

② $J(a,b) = \max(1-a, b)$

③ $J(a,b) = 1-a+ab$

④ $J(a,b) = \begin{cases} a & a=0 \\ 1-a & b=0 \\ 1 & \text{otherwise.} \end{cases}$