Unit - I

☆ Basic Concept :-

1) <u>least upper bound (lub) (suprimum)</u> :-

A real number r is called a least upper bound of a set A if

(i) 'r' is an upper bound of A

(ii) No real number less than r is an upper bound

The lub of A is also denoted by sup A

2) Greatest lower bound (infrimum)

let A be any set of real number if there exist a real number S, such that $S \leq x, \forall x \in A$ then S is called a lower bound of A.

The number x is called greatest lower bound of A if

i) S is lower bound of A

ii) No real number greater than S is the lower bound of A.

The greatest lower bound of A is also denoted by inf A.

* Fuzzy Sets

let X be any set. A fuzzy subset A of X is defined by membership function μ_A

$\mu_A : X \longrightarrow [0;1]$ or $A : X \longrightarrow [0,1]$

• Basic Concept of fuzzy set

Def^n

1) α-cut of fuzzy set

let $A : X \longrightarrow I$, $I = [0,1]$ be a fuzzy set for any $\alpha \in [0,1]$ be defined α-cut of A,

$\boxed{{}^{\alpha}A = \{x \in X \;/\; A(x) \geq \alpha\}}$

② Strong α-cut of set
It is defined as,
$\alpha_A^+ = \{x \in X / A(x) > \alpha\}$

★ Note :-
1) 0-cut of A = $0_A = \{x \in X / A(x) \geq 0\}$

2) 1-cut of A = $1_A = \{x \in X / A(x) \geq 1\}$
i.e, $= \{x \in X / A(x) = 1\}$

3) 1_A is called core of A

3) $0_A^+ = \{x \in X / A(x) > 0\}$

~~strong-0-cut~~
0_A^+ is called support of A

4) $1_A^+ = \{x \in X / A(x) > 1\} = \phi$

5) let $\alpha \leq \beta \Rightarrow \beta_A \subseteq \alpha_A$
let $x \in \beta_A$
$\Rightarrow A(x) \geq \beta$
$\Rightarrow A(x) \geq \beta \geq \alpha$
$\Rightarrow A(x) \geq \alpha$
$\Rightarrow x \in \alpha_A$
$\Rightarrow \beta_A \subseteq \alpha_A$
$\Rightarrow \beta_A \subseteq \alpha_A$ whenever $\alpha \leq \beta$.

6) Support of fuzzy set :- let A be a fuzzy set defined on X. A set of all elements whose membership value is non-zero is called support of A.

$\lambda r + (1-\lambda)s$

Thus, support of A = $\{x \in X \mid A(x) > 0\}$
$= 0_A^+$

7) Core of fuzzy set :-
The set of all element of X whose membership value is one is called the core of fuzzy set.
Thus, core of A = $\{x \in X / A(x) = 1\}$
$= 1_A$

8) Height of fuzzy set :-
The leargest membership value of the element of X is called height of fuzzy set & it is denoted by $h(A)$.
i.e, $h(A) = \sup_{x \in X} A(x)$

If $h(A) = 1$ then A is called normal fuzzy set otherwise it is called subnormal fuzzy set

9) Convex Set :-
A set 'A' in $\mathbb{R}^n$ is called a convex set is for every pair of point $r, s \in A$. The point $\boxed{\lambda r + (1-\lambda)s \in A}$ $\forall \lambda \in [0,1]$
where, $r = (r_1, r_2, \dots, r_n)$ &
$s = (s_1, s_2, \dots, s_n)$ $r_i, s_i \in \mathbb{R}$ $\forall i$
i.e, A is convex set if for any two points $r, s \in A$. the line joining r, s lies entirely in A.

Convex Fuzzy Set :-

let $A : X \to \mathbb{R}$ be a fuzzy set defined on X. the fuzzy set A is called convex fuzzy set. if every α-cut of A is a convex set. $\forall \alpha \in [0,1]$

* Remark

various notation for represention of fuzzy set

1) If A is a fuzzy set defined on finite universal set X then we write,

$$A = \frac{A(x_1)}{x_1} + \frac{A(x_2)}{x_2} + \dots + \frac{A(x_n)}{x_n}$$

$$A = \sum_{n} \frac{A(x_n)}{x_n}$$

the negative sign indicate that, linking of the element of the universal set X and if grade positive indicate togetherness

2) If for a real interval X, the fuzzy set defined on X is denoted by,

$$A = \int_{x} \frac{A(x)}{x}$$

* Example

① For a fuzzy set

$$A(x) = \begin{cases} \frac{5-x}{3} & , 2 \le x \le 5 \\ \frac{x+2}{4} & , -2 \le x \le 2 \\ 0 & , \text{otherwise} \end{cases}$$

find 1_A, 0^+_A, $h(A)$

$\Rightarrow$ Solution

Given,

$$A(x) = \begin{cases} \frac{5-x}{3} & , 2 \le x \le 5 \\ \frac{x+2}{4} & , -2 \le x \le 2 \\ 0 & , \text{otherwise} \end{cases}$$

$${}^{\alpha}A = \{ x \in X / A(x) \ge \alpha \}$$

$\Rightarrow A(x) \ge \alpha$

$\Rightarrow \frac{5-x}{3} \ge \alpha, \quad \frac{x+2}{4} \ge \alpha$

$\Rightarrow 5-x \ge 3\alpha, \quad x+2 \ge 4\alpha$

$\Rightarrow -x \ge 3\alpha - 5, \quad x \ge 4\alpha - 2$

$\Rightarrow x \le 5 - 3\alpha, \quad x \ge 4\alpha - 2$

$\Rightarrow 4\alpha - 2 \le x \le 5 - 3\alpha$

$\Rightarrow [4\alpha - 2, 5 - 3\alpha]$

$1_A = [4-2, 5-3]$

$1_A = [2, 2]$

$0^+_A = [-2, 5]$

$h(A) = 1$

2) If A is a fuzzy set of $\mathbb{R}$ defined by

$$A(x) = \begin{cases} 0 & , x \le -1, x \ge 3 \\ \frac{x+1}{2} & , -1 \le x \le 1 \\ \frac{3-x}{2} & , 1 \le x \le 3 \end{cases}$$

find 1_A, 0^+_A, $h(A)$ (ie, core of A, support of A & height of A)

Solution:

Given

$$A(x) = \begin{cases} 0 & , x \le -1, x \ge 3 \\ \frac{x+1}{2} & , -1 \le x \le 1 \\ \frac{3-x}{2} & , 1 \le x \le 3 \end{cases}$$

${}^{\alpha}A = \{x \in X \mid A(x) \ge \alpha\}$

$\Rightarrow A(x) \ge \alpha$

$\Rightarrow \frac{x+1}{2} \ge \alpha$, $\quad \frac{3-x}{2} \ge \alpha$

$\Rightarrow x+1 \ge 2\alpha \quad\quad 3-x \ge 2\alpha$

$\Rightarrow x \ge 2\alpha-1 \quad\quad -x \ge 2\alpha-3$

$\Rightarrow x \ge 2\alpha-1 \quad\quad x \le 3-2\alpha$

$\Rightarrow 2\alpha-1 \le x \le 3-2\alpha$

$\Rightarrow [2\alpha-1, 3-2\alpha]$

${}^{1}A = [2-1, 3-2]$

${}^{1}A = [1,1] = \{1\}$

${}^{0+}A = (-1,3)$

$h(A) = 1$

5) Let A be a fuzzy set defined on $x = \{1, 2, ..., 100\}$ by

$$A(x) = \begin{cases} 1 & , \text{if } x \le 20 \\ \frac{35-x}{15} & , 20 < x < 35 \\ 0 & , x \ge 35 \end{cases}$$

Find ${}^{1}A$, ${}^{0+}A$, $h(A)$ & ${}^{(0.5)}A$

Solution:

By def^n of α-cut of A

${}^{\alpha}A = \{x \in X \mid A(x) \ge \alpha\}$

$\Rightarrow A(x) \ge \alpha$

Now,

support of A = ${}^{0+}A = \{x \in X \mid A(x) > 0\}$

${}^{0+}A = \{1, 2, ..., 34\}$

core of A = ${}^{1}A = \{x \in X \mid A(x) \ge 1\}$

${}^{1}A = \{1, 2, ..., 20\}$

height of A = $h(A) = 1$.

${}^{(0.5)}A = \{x \in X \mid A(x) \ge \frac{1}{2}\}$

$\Rightarrow A(x) \ge \frac{1}{2}$

$\Rightarrow \frac{35-x}{15} \ge \frac{1}{2}$

$\Rightarrow 2(35-x) \ge 15$

$\Rightarrow 70-2x \ge 15$

$-2x \ge 15-70$

$2x \le 70-15$

$2x \le 55$

$x \le \frac{55}{2}$

$x \le 27.5$

${}^{0.5}A = \{1, 2, 3, ..., 27\}$

4) For a fuzzy set defined on $x = \{0,1,2,\dots,100\}$ by.

$$A(x) = \begin{cases} 0 , & \text{if } x \le 45 \\ \frac{x-45}{15} , & \text{if } 45 \le x \le 66 \\ 1 , & \text{if } x \ge 60 \end{cases}$$

Find support of A, 1_A, $h(A)$, 0.8_A

Solution:

Given,

$$A(x) = \begin{cases} 0 , & \text{if } x \le 45 \\ \frac{x-45}{15} , & \text{if } 45 \le x \le 66 \\ 1 , & \text{if } x \ge 60 \end{cases}$$

By def^n of α-cut of A

$\alpha_A = \{x \in X \,/\, A(x) \ge \alpha\}$

$\Rightarrow A(x) \ge \alpha$

Now,

support of A = $0_A^+ = \{x \in X \,/\, A(x) > 0\}$

$0_A^+ = \{46, 47, \dots, 100\}$

core of A, = $1_A = \{x \in X \,/\, A(x) \ge 1\}$

$1_A = \{60, 61, \dots, 100\}$

height of A = $h(A) = 1$

$(0.8)_A = \{x \in X \,/\, A(x) \ge 0.8\}$

$A(x) \ge 0.8$

$\frac{x-45}{15} \ge 0.8$

$x - 45 \ge 12$

$x \ge 12 + 45$

$x \ge 57$

$0.8_A = \{57, 58, \dots, 100\}$

5) let M be a fuzzy set defined on

$x = \{0,1,2,\dots,100\}$ by;

$$M(x) = \begin{cases} 0 , & \text{if } x \le 20,\ x \ge 60 \\ \frac{x-20}{15} , & \text{if } 20 \le x \le 35 \\ \frac{60-x}{15} , & \text{if } 45 \le x \le 60 \\ 1 , & \text{if } 35 \le x \le 45 \end{cases}$$

Find 1_M, 0.5_M, 0.8_M, 0_M^+.

Sol^n:-

By def^n of α-cut of A

$\alpha_A = \{x \in X \,/\, A(x) \ge \alpha\}$

~~$\Rightarrow A(x) \ge \alpha$~~

$\Rightarrow A(x) \ge \alpha$

Now, $0_M^+ = \{x \in X \,/\, M(x) > 0\}$

$^{1}M = \{x \in X \mid M(x) \geq 1\}$
$= \{35, 36, \ldots, 45\}$

$^{0.5}M = \{x \in X \mid M(x) \geq 0.5\}$

$M(x) \geq 0.5$

$\frac{x-20}{15} \geq \frac{1}{2}$ $\qquad$ $\frac{60-x}{15} \geq \frac{1}{2}$

$x - 20 \geq \frac{15}{2}$ $\qquad$ $60 - x \geq \frac{15}{2}$

$x \geq \frac{15}{2} + 20$ $\qquad$ $-x \geq \frac{15}{2} - 60$

$x \geq \frac{55}{2}$ $\qquad$ $x \leq 60 - \frac{15}{2}$

$x \geq 27.5$ $\qquad$ $x \leq \frac{105}{2}$

$x \leq 52.5$

$27.5 \leq x \leq 52.5$
$^{0.5}M = \{28, 29, \ldots, 52\}$

$^{0.8}M = \{x \in X \mid M(x) \geq 0.8\}$
$M(x) \geq 0.8$

$\frac{x-20}{15} \geq 0.8$ $\qquad$ $\frac{60-x}{15} \geq 0.8$

$x - 20 \geq 12$ $\qquad$ $60 - x \geq 12$

$x \geq 12 + 20$ $\qquad$ $-x \geq 12 - 60$

$x \geq 32$ $\qquad$ $\Rightarrow x \leq 60 - 12$

$x \leq 48$

$32 \leq x \leq 48$
$^{0.8}M = \{32, 33, \ldots, 48\}$
$^{0+}M = \{x \in X \mid A(x) > 0\}$
$= \{21, 22, \ldots, 59\}$

Theorem :-

Statement : A fuzzy set 'A' on R is a convex iff $A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\}$ $\forall x_1, x_2$

proof :
let A be a convex fuzzy set
$\therefore$ $^{\alpha}A$ is convex set, $\forall \alpha \in (0,1]$
let $x_1, x_2 \in R$ s.t.
$A(x_1) \leq A(x_2)$
let $A(x_1) = \alpha$
$\therefore$ ~~$A(x_1) \geq \alpha$ and~~
$A(x_2) \geq A(x_1) = \alpha$
$\Rightarrow A(x_1) \geq \alpha$ & $A(x_2) \geq A(x_1) = \alpha$
$\Rightarrow$ ~~$A(x_2)$~~ $x_1, x_2 \in {}^{\alpha}A$
but $^{\alpha}A$ is a convex set
$\therefore$ $\lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A$
$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = A(x_1)$
$\Rightarrow A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\}$
$\Rightarrow A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\}$

Conversely,
the fuzzy set A on R, satisfy the condⁿ,
$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\}$ $\forall x_1, x_2$
$\lambda \in [0,1]$

we have to prove that, the set A is a convex fuzzy set.
ie, to prove that, $^{\alpha}A$ is convex set
let $\alpha \in [0,1]$ be any real number consider
$^{\alpha}A$ α-cut of A, $\forall \alpha > 0$, given by,
$^{\alpha}A = \{x \in R \mid A(x) \geq \alpha\}$
let $x_1, x_2 \in {}^{\alpha}A$
$\Rightarrow A(x_1) \geq \alpha$ & $A(x_2) \geq \alpha$
for any $\lambda \in [0,1]$, we have,

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$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{A(x_1), A(x_2)\}$$
$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\alpha, \alpha\}$$
$$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha \quad \forall \lambda \in [0,1]$$
$$\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A$$

this shows that ${}^{\alpha}A$ is convex set

since λ is arbitrary

$\therefore$ every α-cut of A is convex set.

Hence The Fuzzy set A is a convex set.

Hence the proof.

* Standard operation on fuzzy set

1) Intersection or Fuzzy intersection :-

let A & B be two fuzzy sets defined on X then $A \cap B : X \to I$ defined by

$$(A \cap B)(x) = \min\{A(x), B(x)\}$$
$$= A(x) \wedge B(x)$$

2) Fuzzy Union :-

let A & B be two fuzzy sets defined on X. then $A \cup B : X \to I$ defined by

$$(A \cup B)(x) = \max\{A(x), B(x)\}$$
$$= A(x) \vee B(x)$$

* Example :-

1) let $A = \frac{0.2}{x_1} + \frac{0.3}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$

$B = \frac{0}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{0.9}{x_5}$

be two fuzzy set, find intersection & union.

$$\Rightarrow A \cap B = \min\{A, B\}$$
$$= \frac{0}{x_1} + \frac{0.3}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{0.9}{x_5}$$

$$A \cup B = \max\{A, B\}$$
$$= \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

3) Fuzzy Complement :

let A be a fuzzy set defined on X then $\bar{A} : X \to I$ defined by,

$$\boxed{\bar{A}(x) = 1 - A(x)}$$

$\bar{A}$ is Fuzzy complement of A.

* Example

let $A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{1}{x_5}$ be a fuzzy set then find fuzzy complement.

$$\Rightarrow \bar{A} = 1 - A$$
$$= \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.3}{x_4} + \frac{0}{x_5}$$

* Note :

Law of contraction and law of excluded middle are not true for the fuzzy sets. $A \cap \bar{A} \neq \phi$, $A \cup \bar{A} \neq X$.

i.e., Law of contraction : $A \cap \bar{A} \neq \phi$

law of excluded middle: $A \cup \bar{A} \neq X$

where ϕ and $\phi : X \to I$ such that,

$\phi(x) = 0, \ \forall x$ & $X : X \to I$ such that,

$X(x) = 1$

* Scalar Cardinality :

For a fuzzy set A defined on a finite universal set X we define a scalar cardinality of A by,

$$\boxed{|A| = \sum_{x \in X} A(x)}$$

Example

1) let $A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.7}{x_4} + \frac{1}{x_5}$

find scalar cardinality.

⇒ by defⁿ of scalar cardinality,

$$|A| = \sum_{x \in X} A(x)$$

$$= A(x_1) + A(x_2) + A(x_3) + A(x_4) + A(x_5)$$

$$= 0.2 + 0.4 + 0.5 + 0.7 + 1$$

$$= 2.8$$

2) Compute the scalar cardinality for the following fuzzy set define on $X = \{0,1,2,3\}$ by $A(x) = \frac{1}{2^x}$, $\forall x \in X$

⇒ Solution :

Here, $A(x) = \frac{1}{2^x}$, $\cancel{\forall x \in X}$ $X = \{0,1,2,3\}$

$A(0) = \frac{1}{2^0} = 1$

$A(1) = \frac{1}{2^1} = \frac{1}{2} = 0.5$

$A(2) = \frac{1}{2^2} = \frac{1}{4} = 0.25$

$A(3) = \frac{1}{2^3} = \frac{1}{8} = 0.125$

The scalar cardinality of A is given by,

$$|A| = \sum_{x \in X} A(x)$$

$$= \sum_{x \in X} A(0) + A(1) + A(2) + A(3)$$

$$= 1 + 0.5 + 0.25 + 0.125$$

$$= 1.875$$

3) Compute the scalar cardinality for the following fuzzy set

① $B(x) = \frac{x}{x+3}$, $X = \{0,1,2,3,4,5\}$

② $D(x) = \frac{x}{x+1}$, $X = \{0,1,2,3,\dots,10\}$

⇒ Solution ①

here, $B(x) = \frac{x}{x+3}$, $X = \{0,1,2,3,4,5\}$

$B(0) = \frac{0}{0+3} = 0$, $B(3) = \frac{3}{3+3} = \frac{3}{6} = 0.5$

$B(1) = \frac{1}{1+3} = \frac{1}{4} = 0.25$ $B(4) = \frac{4}{4+3} = \frac{4}{7} = 0.571$

$B(2) = \frac{2}{2+3} = \frac{2}{5} = 0.4$ $B(5) = \frac{5}{5+3} = \frac{5}{8} = 0.625$

0.25
0.4
0.5
0.57
0.625
2.346

The scalar cardinality of B is given by,

$$|B| = \sum_{x \in X} B(x)$$

$$= 0 + 0.25 + 0.4 + 0.5 + 0.571 + 0.625$$

$$= 2.3464$$

Solution ②

$D(x) = \frac{x}{x+1}$, $X = \{0,1,2,3,\dots,10\}$

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$D(0) = \frac{0}{0+1} = 0$ $\quad$ $D(6) = \frac{6}{6+1} = \frac{6}{7} = 0.851$

$D(1) = \frac{1}{1+1} = \frac{1}{2} = 0.5$ $\quad$ $D(7) = \frac{7}{7+1} = \frac{7}{8} = 0.875$

$D(2) = \frac{2}{2+1} = \frac{2}{3} = 0.66$ $\quad$ $D(8) = \frac{8}{8+1} = \frac{8}{9} = 0.888$

$D(3) = \frac{3}{3+1} = \frac{3}{4} = 0.75$ $\quad$ $D(9) = \frac{9}{9+1} = \frac{9}{10} = 0.90$

$D(4) = \frac{4}{4+1} = \frac{4}{5} = 0.8$ $\quad$ $D(10) = \frac{10}{10+1} = \frac{10}{11} = 0.909$

$D(5) = \frac{5}{5+1} = \frac{5}{6} = 0.83$

The scalar cardinality of D is given by

$$|D| = \sum_{x \in X} D(x)$$

$$= D(0) + D(1) + D(2) + D(3) + D(4) + D(5) + D(6) + D(7) + D(8) + D(9) + D(10)$$

$$= 0 + 0.5 + 0.66 + 0.75 + 0.8 + 0.83 + 0.851 + 0.875 + 0.888 + 0.90 + 0.909$$

$$|D| = 7.963$$

4) Ex. If A & B be any two fuzzy set defined by $A = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.9}{x_5}$

& $B = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{1}{x_4} + \frac{0.1}{x_5}$

then find (i) $A \cup B$ (ii) $A \cap B$ (iii) $\bar{B}$ (iv) $|A|$ (v) $\bar{A}$ (vi) $\overline{A \cup B}$ (vii) $\overline{A \cap B}$ (viii) $\bar{A} \cap \bar{B}$

$\Rightarrow$ Given,

$A = \frac{0.3}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{0.2}{x_4} + \frac{0.9}{x_5}$

& $B = \frac{0.6}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{1}{x_4} + \frac{0.1}{x_5}$

i) $A \cap B = \min\{A, B\}$

$= \frac{0.3}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.1}{x_5}$

ii) $A \cup B = \max\{A, B\}$

$= \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.7}{x_3} + \frac{1}{x_4} + \frac{0.9}{x_5}$

iii) $\bar{B} \Rightarrow 1 - B$

$= \frac{0.4}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0}{x_4} + \frac{0.9}{x_5}$

iv) $\bar{A} \Rightarrow 1 - A$

$= \frac{0.7}{x_1} + \frac{0.4}{x_2} + \frac{0.3}{x_3} + \frac{0.8}{x_4} + \frac{0.1}{x_5}$

v) $|A| \Rightarrow \sum_{x \in X} A(x)$

$= A(x_1) + A(x_2) + A(x_3) + A(x_4) + A(x_5)$

$= 0.3 + 0.6 + 0.7 + 0.2 + 0.9$

$= \cancel{2.9}\ 2.7$

vi) $\overline{A \cup B} = 1 - A \cup B$

$= \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.3}{x_3} + \frac{0}{x_4} + \frac{0.1}{x_5}$

vii) $\overline{A \cap B} = 1 - A \cap B$

$= \frac{0.7}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.8}{x_4} + \frac{0.9}{x_5}$

viii) $\bar{A} \cap \bar{B} = \min\{\bar{A}, \bar{B}\}$

$= \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.3}{x_3} + \frac{0}{x_4} + \frac{0.1}{x_5}$

* Law of absorption

Theorem :-

4 mark For any fuzzy set A & B on X .

i) $A \cup (A \cap B) = A$

ii) $A \cap (A \cup B) = A$

proof :-

For $x \in X$

$(A \cup (A \cap B))(x) = \max\{A(x), A \cap B(x)\}$

$= A(x) \vee A \cap B(x)$

$(A \cup (A \cap B))(x) = A(x) \vee [A(x) \wedge B(x)]$

Now $A(x)$ and $B(x)$ are the elements of I which is totally ordered set hence either $A(x) \leq B(x)$ or $A(x) \geq B(x)$

Case I:

If $A(x) \leq B(x)$ then

$[A \cup (A \cap B)](x) = A(x) \vee [A(x) \wedge B(x)]$

$= A(x) \vee A(x)$

$= A(x)$

$[A \cup (A \cap B)](x) = A(x)$

thus, $A \cup (A \cap B) = A$

Case II :

If $A(x) \geq B(x)$ then

$[A \cup (A \cap B)](x) = A(x) \vee [A(x) \wedge B(x)]$

$= A(x) \vee B(x)$

$[A \cup (A \cap B)](x) = A(x)$

Thus, $A \cup (A \cap B) = A$

Next, for $x \in X$

$(A \cap (A \cup B))(x) = \min\{A(x), A \cup B(x)\}$

$= A(x) \wedge A \cup B(x)$

$(A \cap (A \cup B))(x) = A(x) \wedge [A(x) \vee B(x)]$

Now $A(x)$ & $B(x)$ are the elements of I which is totally ordered set.

hence either $A(x) \geq B(x)$ or $A(x) \leq B(x)$

Case I :

If $A(x) \geq B(x)$ then

$[A \cap (A \cup B)(x)] = A(x) \wedge [A(x) \vee B(x)]$

$= A(x) \wedge [A(x)]$

$\therefore [A \cap (A \cup B)(x)] = A(x)$

$A \cap (A \cup B) = A$

Case II :

If $A(x) \leq B(x)$

$[A \cap (A \cup B)](x) = A(x) \wedge [A(x) \vee B(x)]$

$= A(x) \wedge [B(x)]$

$[A \cap (A \cup B)](x) = A(x)$

$A \cap (A \cup B) = A$

Thus for any $x \in X$, we have

$A \cap (A \cup B) = A$

* Fuzzy Subset

let A, B be the fuzzy set defined on X we say that $A \subseteq B$ if $A(x) \leq B(x)$, $\forall x \in X$ ~~fr~~.

* Example

1) $A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.3}{x_3}$ & $B = \frac{0.4}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3}$

$A(x) \leq B(x)$

i.e. $A \subseteq B$

Q) ~~show that $A \subseteq B$ iff~~

Th^m 2) Show that, $A \subseteq B$ iff $A \cap B = A$ & $A \cup B = B$ where, A & B are fuzzy set. Defined ~~the~~ on X.

proof:-

let $A \subseteq B$

$\Rightarrow A(x) \leq B(x), \ \forall x \in X$

where A & B are fuzzy set defined on X.

Now,

$(A \cap B)(x) = A(x) \wedge B(x)$

$(A \cap B)(x) = A(x)$

$\Rightarrow A \cap B = A$

and $(A \cup B)(x) = A(x) \vee B(x)$

$(A \cup B)(x) = B(x)$

$(A \cup B) = B$

Conversely,

$A \cap B = A$ & $A \cup B = B$ holds true.

then for $x \in X$, ~~$A \cup B(x)$~~

$(A \cup B)(x) = B(x)$

$\Rightarrow A(x) \vee B(x) = B(x)$

$\Rightarrow A(x) \leq B(x) \quad \forall x \in X$ —— ①

$(A \cap B)(x) = A(x)$

$A(x) \wedge B(x) = A(x)$

$\Rightarrow A(x) \leq B(x)$ —— ②

from eq^n ① & ②

we have,

$A(x) \leq B(x) \quad \forall x \in X$

$\Rightarrow A \subseteq B$

Hence the proof.

* Theorem

If A & B are fuzzy set, defined on X then show that

$|A| + |B| = |A \cup B| + |A \cap B|$

proof:-

for any $x \in X$,

$A(x), B(x) \in I$

since I is totally ordered set.

either $A(x) \leq B(x)$ or $A(x) \geq B(x)$

Case I:

If $A(x) \leq B(x)$ then

$A(x) \vee B(x) = B(x)$

& $A(x) \wedge B(x) = A(x)$

$[A(x) \vee B(x)] + [A(x) \wedge B(x)] = B(x) + A(x)$ —— ①

Case II:

If $A(x) \geq B(x)$ then

$A(x) \vee B(x) = A(x)$

& $A(x) \wedge B(x) = B(x)$

$[A(x) \vee B(x)] + [A(x) \wedge B(x)] = A(x) + B(x)$ —— ②

Thus, $\forall x \in X$, we have,

$[A(x) \vee B(x)] + [A(x) \wedge B(x)] = A(x) + B(x)$

$\Rightarrow [(A \cup B)(x)] + [(A \cap B)(x)] = A(x) + B(x) \quad \forall x \in X$

$\Rightarrow$

$\Rightarrow \sum_{x \in X} (A \cup B)(x) + \sum_{x \in X} (A \cap B)(x) = \sum_{x \in X} A(x) + \sum_{x \in X} B(x)$

$\Rightarrow |A \cup B| + |A \cap B| = |A| + |B|$

$\Rightarrow$ i.e, $|A| + |B| = |A \cup B| + |A \cap B|$

Hence the proof.

* Degree of subsethood

let A & B be two fuzzy set, defined on X. the degree of subsethood of A in B is denoted by S(A,B) is defined as,

$$\boxed{S(A,B) = \frac{|A \cap B|}{|A|}}$$

• Note

$S(B,A) = \frac{|A \cap B|}{|B|}$ (if B in A)

* Example

1) If A & B are fuzzy set, defined by

$A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$

& $B = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.9}{x_4} + \frac{0.1}{x_5}$ then.

find S(A,B) & S(B,A)

⟹ Solution:

Here,

$A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$

& $B = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.9}{x_4} + \frac{0.1}{x_5}$

$S(A,B) = \frac{|A \cap B|}{|A|}$

$A \cap B = \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.8}{x_4} + \frac{0.1}{x_5}$

$|A \cap B| = \sum_{x \in X} A(x)$

$= \cancel{A(x_1)} + \cancel{A(x_2)} + \cancel{A(x_3)} + A(x_4) + A(x_5)$

$= 0.2 + 0.2 + 0.5 + 0.8 + 0.1$

$= 1.8$

$|A| = \sum_{x \in X} A(x)$

$= A(x_1) + A(x_2) + A(x_3) + A(x_4) + A(x_5)$

$= 0.2 + 0.4 + 0.6 + 0.8 + 1$

$= 3$

$S(A,B) = \frac{|A \cap B|}{|A|}$

$S(A,B) = \frac{1.8}{3}$

$S(A,B) = 0.6$

$|B| = \sum_{x \in X} B(x)$

$= B(x_1) + B(x_2) + B(x_3) + B(x_4) + B(x_5)$

$= 0.4 + 0.2 + 0.5 + 0.9 + 0.1$

$= 2.1$

$S(B,A) = \frac{|A \cap B|}{|B|}$

$S(B,A) = \frac{1.8}{2.1}$

$S(B,A) = 0.8571$

2) If A & B are fuzzy set, defined by,

$A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$ &

$B = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.9}{x_4} + \frac{0.1}{x_5}$

Determine the following fuzzy set.

1) $\bar{A}$, 2) $\bar{B}$, 3) $A \cup B$, 4) $A \cap B$, 5) $A \cup \bar{A}$ 6) $A \cup B$

1) $\bar{B}$, 2) $\bar{A}$ 3) $A \cup B$ 4) $A \cap B$ 5) $A \cup \bar{A}$
6) $\overline{A \cup B}$ 7) $A \cap \bar{A}$ 8) $\overline{A \cap B}$ 9) $\bar{A} \cap B$
10) $A \cap \bar{B}$ 11) $\bar{A} \cap \bar{B}$ 12) $\bar{A} \cup B$ 13) $\bar{A} \cup \bar{B}$ 14) $A \cup \bar{B}$
15) $B \cup \bar{B}$ 16) $S(A,B)$, 17) $S(B,A)$ 18)
18) $A \Delta B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$ 19) $|A|$ 20) $|B|$ 21) $|A \cup B|$
⇒ 22) $|A \cap B|$.

Given,

$$A = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

$$B = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.9}{x_4} + \frac{0.1}{x_5}$$

1] $\bar{B} \Rightarrow \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.1}{x_4} + \frac{0.9}{x_5}$

2] $\bar{A} \Rightarrow \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0}{x_5}$

3] $A \cup B \Rightarrow \max\{A, B\}$

$$\frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.9}{x_4} + \frac{1}{x_5}$$

4) $A \cap B \Rightarrow \min\{A, B\}$

$$\frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.5}{x_3} + \frac{0.8}{x_4} + \frac{0.1}{x_5}$$

5) $A \cup \bar{A} \Rightarrow \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$

6) $\overline{A \cup B} \Rightarrow \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0}{x_5}$

7) $A \cap \bar{A} \Rightarrow \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0}{x_5}$

8) $\overline{A \cap B} = \frac{0.8}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.9}{x_5}$

9) $\bar{A} \cap B = \frac{0.4}{x_1} + \frac{0.2}{x_2} + \frac{0.4}{x_3} + \frac{0.2}{x_4} + \frac{0}{x_5}$

10) $A \cap \bar{B} = \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.1}{x_4} + \frac{0.9}{x_5}$

11) $\bar{A} \cap \bar{B} = \frac{0.6}{x_1} + \frac{0.6}{x_2} + \frac{0.4}{x_3} + \frac{0.1}{x_4} + \frac{0}{x_5}$

12) $\bar{A} \cup B = \frac{0.8}{x_1} + \frac{0.6}{x_2} + \frac{0.5}{x_3} + \frac{0.9}{x_4} + \frac{0.1}{x_5}$

13) $\bar{A} \cup \bar{B} = \frac{0.8}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.9}{x_5}$

14) $A \cup \bar{B} = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$

15) $B \cup \bar{B} = \frac{0.6}{x_1} + \frac{0.8}{x_2} + \frac{0.5}{x_3} + \frac{0.9}{x_4} + \frac{0.9}{x_5}$

16) $S(A,B) = \frac{|A \cap B|}{|A|}$

$$= \frac{1.8}{3}$$

$$= 0.6$$

17) $S(B,A) = \frac{|A \cap B|}{|B|}$

$$= \frac{1.8}{2.1}$$

$$= 0.8571$$

18) $A \Delta B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$

$= \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.5}{x_3} + \frac{0.2}{x_4} + \frac{0.9}{x_5}$

19) $|A| = \sum_{x \in X} A(x)$

$= A(x_1) + A(x_2) + A(x_3) + A(x_4) + A(x_5)$

$= 0.2 + 0.4 + 0.6 + 0.8 + 1$

$= 3$

20) $|B| = \sum_{x \in X} B(x)$

$= B(x_1) + B(x_2) + B(x_3) + B(x_4) + B(x_5)$

$= 0.4 + 0.2 + 0.5 + 0.9 + 0.1$

$= 2.1$

21) $|A \cup B| = \sum_{x \in X} (A \cup B)(x)$

$= 0.4 + 0.4 + 0.6 + 0.9 + 1$

$= 2.4$

22) $|A \cap B| = \sum_{x \in X} (A \cap B)(x)$

$= 0.2 + 0.2 + 0.5 + 0.8 + 0.1$

$= 0.5 + 0.5 + 0.8$

$= 1.8$

* Note :-

For the fuzzy set, D-Morgan's law holds true,

ie, $\overline{A \cup B} = \bar{A} \cap \bar{B}$ & $\overline{A \cap B} = \bar{A} \cup \bar{B}$

1) $= 1 - (A \cup B)(x)$

$= 1 - \max\{A(x), B(x)\}$

$= \min\{1 - A(x), 1 - B(x)\}$

$= \min\{\bar{A}(x), \bar{B}(x)\}$

$= (\bar{A} \cap \bar{B})(x)$

$= \bar{A} \cap \bar{B}$

2) $\overline{A \cap B}(x) = (1 - A \cap B)(x)$

$= 1 - \min\{A(x), B(x)\}$

$= \max\{1 - A(x), 1 - B(x)\}$

$= \max\{\bar{A}(x), \bar{B}(x)\}$

$= (\bar{A} \cup \bar{B})(x)$

Ex. Determine the degree of subsethood $S(A,B)$ & $S(B,A)$ for the fuzzy set

$A(x) = \frac{2x}{x+5}$, $B(x) = \frac{1-x}{5}$

for $x = \{0, 1, 2, 3, 4, 5\}$

$\Rightarrow$ here,

$A(x) = \frac{2x}{x+5}$, $B(x) = \frac{1-x}{5}$

$x = 0$, $A(0) = \frac{2(0)}{0+5} = 0$

$x = 1$, $A(1) = \frac{2(1)}{1+5} = \frac{2}{6} = \frac{1}{3} = 0.33$

$x = 2$, $A(2) = \frac{2(2)}{2+5} = \frac{4}{7} = 0.571$

$x = 3$, $A(3) = \frac{2(3)}{3+5} = \frac{6}{8} = 0.75$

$x = 4$, $A(4) = \frac{2(4)}{4+5} = \frac{8}{9} = 0.88$

$x = 5$, $A(5) = \frac{2(5)}{5+5} = \frac{10}{10} = 1$

$x = 0$, $B(0) = \frac{1-0}{5} = 1$

$x = 1$, $B(1) = 1 - \frac{1}{5} = 1 - 0.2 = 0.8$

$x = 2$, $B(2) = 1 - \frac{2}{5} = 1 - 0.4 = 0.6$

$x = 3$, $B(3) = 1 - \frac{3}{5} = 1 - 0.6 = 0.4$

$x = 4$, $B(4) = 1 - \frac{4}{5} = 0.2$

$x = 5$, $B(5) = 1 - \frac{5}{5} = 0$

$A = \frac{0}{x_1} + \frac{0.33}{x_2} + \frac{0.57}{x_3} + \frac{0.75}{x_4} + \frac{0.88}{x_5} + \frac{1}{x_6}$

$B = \frac{1}{x_1} + \frac{0.8}{x_2} + \frac{0.6}{x_3} + \frac{0.4}{x_4} + \frac{0.2}{x_5} + \frac{0}{x_6}$

$A \cap B = \min\{A(x), B(x)\}$

$= \frac{0}{x_1} + \frac{0.33}{x_2} + \frac{0.571}{x_3} + \frac{0.4}{x_4} + \frac{0.2}{x_5} + \frac{0}{x_6}$

$|A \cap B| = 0 + 0.33 + 0.57 + 0.4 + 0.2 + 0$

$= 1.501$

$|A| = \sum_{x \in X} A(x) = 0 + 0.33 + 0.57 + 0.731 + 0.88 + 1$

$= \cancel{3.531}$

$= 3.531$

$S(A,B) = \frac{|A \cap B|}{|A|}$

$= \frac{1.501}{3.531}$

$|B| = 1 + 0.8 + 0.6 + 0.4 + 0.2 + 0$

$= 3$

$S(B,A) = \frac{|A \cap B|}{|B|}$

$= \frac{1.5}{3}$

Theorem :-

Let A & B be the fuzzy set defined on X. Then for any two real no. $\alpha, \beta \in [0,1]$ we have.

1. $\alpha_A^+ \subseteq \alpha_A$
2. $\alpha \le \beta \Rightarrow \beta_A \subseteq \alpha_A$ & $\beta_A^+ \subseteq \alpha_A^+$
3. $\alpha_{A \cap B} = \alpha_A \cap \alpha_B$
4. $\alpha^+_{A \cap B} = \alpha_A^+ \cap \alpha_B^+$
5. $\alpha_{A \cup B} = \alpha_A \cup \alpha_B$
6. $\alpha^+_{A \cup B} = \alpha_A^+ \cup \alpha_B^+$
7. $\alpha_{\bar{A}} = (\overline{(1-\alpha)_A^+})$

Proof:

1. let $x \in X$

let $x \in \alpha_A^+ \Rightarrow A(x) > \alpha$

$\Rightarrow A(x) \ge \alpha$

$\Rightarrow x \in \alpha_A$

Thus $x \in \alpha_A^+ \Rightarrow x \in \alpha_A$

Hence $\alpha_A^+ \subseteq \alpha_A$.

2. let $\alpha \le \beta$, $x \in X$

let $x \in \beta_A \Rightarrow A(x) \ge \beta$

$\Rightarrow A(x) \ge \beta \ge \alpha$

$\Rightarrow A(x) \ge \alpha$

$\Rightarrow x \in \alpha_A$

Thus $x \in \beta_A \Rightarrow x \in \alpha_A$

$\beta_A \subseteq \alpha_A$

Similarly, we have.

$x \in \beta_A^+ \Rightarrow A(x) > \beta$

$\Rightarrow A(x) > \beta \ge \alpha$

$\Rightarrow A(x) > \alpha$

$\Rightarrow x \in \alpha_A^+$

Thus $x \in \beta_A^+ \Rightarrow x \in \alpha_A^+$

Hence, $\beta_A^+ \subseteq \alpha_A^+$

3. let $x \in X$;

let $x \in \alpha_{A \cap B} \Leftrightarrow A \cap B(x) \ge \alpha$

$\Leftrightarrow A(x) \wedge B(x) \ge \alpha$

$\Leftrightarrow A(x) \ge \alpha$ & $B(x) \ge \alpha$

$\Leftrightarrow x \in \alpha_A$ & $x \in \alpha_B$

$\Leftrightarrow x \in \alpha_A \cap \alpha_B$

Thus $x \in \alpha_{A \cap B} \Leftrightarrow x \in \alpha_A \cap \alpha_B$

Hence, $\alpha_{A \cap B} = \alpha_A \cap \alpha_B$.

4) $x \in \alpha^+_{A\cap B} \Leftrightarrow A\cap B(x) > \alpha$
$\Leftrightarrow A(x) \wedge B(x) > \alpha$
$\Leftrightarrow A(x) > \alpha$ & $B(x) > \alpha$
$\Leftrightarrow x \in \alpha^+_A$ & $x \in \alpha^+_B$
$\Leftrightarrow x \in \alpha^+_A \cap \alpha^+_B$

Thus, $x \in \alpha^+_{A\cap B} \Leftrightarrow x \in \alpha^+_A \cap \alpha^+_B$
i.e, $\alpha^+_{A\cap B} = \alpha^+_A \cap \alpha^+_B$

5) $x \in \alpha_{A\cup B} \Leftrightarrow (A\cup B)x \geq \alpha$
$\Leftrightarrow A(x) \vee B(x) \geq \alpha$
$\Leftrightarrow A(x) \geq \alpha$ or $B(x) \geq \alpha$
$\Leftrightarrow x \in \alpha_A$ or $x \in \alpha_B$
$\Leftrightarrow x \in \alpha_A \cup \alpha_B$

Thus, $x \in \alpha_{A\cup B} \Leftrightarrow x \in \alpha_A \cup \alpha_B$
i.e, $\alpha_{A\cup B} = \alpha_A \cup \alpha_B$

6) $x \in \alpha^+_{A\cup B} \Leftrightarrow (A\cup B)x > \alpha$
$\Leftrightarrow A(x) \vee B(x) > \alpha$
$\Leftrightarrow A(x) > \alpha$ or $B(x) > \alpha$
$\Leftrightarrow x \in \alpha^+_A$ or $x \in \alpha^+_B$
$\Leftrightarrow x \in \alpha^+_A \cup \alpha^+_B$

Thus, $x \in \alpha^+_{A\cup B} \Leftrightarrow x \in \alpha^+_A \cup \alpha^+_B$
i.e, $\alpha^+_{A\cup B} = \alpha^+_A \cup \alpha^+_B$

7) let $x \in X$
let $x \in \alpha_{\bar{A}}$
$\Leftrightarrow \bar{A}(x) \geq \alpha$
$\Leftrightarrow (1 - A(x)) \geq \alpha$
$\Leftrightarrow 1 - \alpha \geq A(x)$

$\Leftrightarrow A(x) \leq 1 - \alpha$
$\Leftrightarrow A(x) \not> 1 - \alpha$
$\Leftrightarrow x \notin (1-\alpha)^+_A$
$\Leftrightarrow x \in \overline{(1-\alpha)^+_A}$
i.e, $\alpha_{\bar{A}} = \overline{(1-\alpha)^+_A}$

* Theorem 3:-

Statement : Let A & B be the fuzzy set defined on X then.

1) $A \subseteq B$ iff $\alpha_A \subseteq \alpha_B$, $\forall\ \alpha \in [0,1]$
$A \subseteq B$ iff $\alpha^+_A \subseteq \alpha^+_B$ $\forall\ \alpha \in [0,1]$

2) $A = B$ iff $\alpha_A = \alpha_B$ $\forall\ \alpha \in [0,1]$
& $A = B$ iff $\alpha^+_A = \alpha^+_B$ $\forall\ \alpha \in [0,1]$

proof :-
let A & B be the fuzzy set defined on X.

1) $A \subseteq B$ then $A(x) \leq B(x)$, $\forall\ x \in X$
Now, for any $\alpha \in [0,1]$
let $x \in \alpha_A \Rightarrow A(x) \geq \alpha$
$\Rightarrow B(x) \geq A(x) \geq \alpha$
$\Rightarrow B(x) \geq \alpha$
$\Rightarrow x \in \alpha_B$
$x \in \alpha_A \Rightarrow x \in \alpha_B$
i.e, $\alpha_A \subseteq \alpha_B$

Conversely,
let $\alpha_A \subseteq \alpha_B$ $\forall\ \alpha \in [0,1]$
for any $x \in X$
let $A(x) = \alpha_0 \in I$
Then,
$A(x) = \alpha_0 \Rightarrow A(x) \geq \alpha_0$
$\Rightarrow x \in \alpha_{0A}$

$\Rightarrow x \in \alpha_{0A} \subseteq \alpha_{0B}$

$\Rightarrow x \in \alpha_{0B}$

$\Rightarrow B(x) \geq \alpha_0$

$\Rightarrow B(x) \geq A(x)$

$\Rightarrow A(x) \leq B(x)$

i.e, $A \subseteq B$

Thus, $A \subseteq B$ iff $\alpha_A \subseteq \alpha_B$, $\forall\ \alpha \in [0,1]$

let $A \subseteq B$ then $A(x) \leq B(x)$, $\forall\ x \in X$

For any $\alpha \in [0,1]$

$x \in \alpha_A^+ \Rightarrow A(x) > \alpha$

$\Rightarrow B(x) \geq A(x) > \alpha$

$\Rightarrow B(x) > \alpha$

$\Rightarrow x \in \alpha_B^+$

$x \in \alpha_A^+ \Rightarrow x \in \alpha_B^+$

i.e, $\alpha_A^+ \subseteq \alpha_B^+$

Conversely,

let $\alpha_A^+ \subseteq \alpha_B^+$, $\forall\ \alpha \in [0,1]$

For any $x \in X$, we have to show that

$A \subseteq B$

i.e, $A(x) \leq B(x)$, $\forall\ x \in X$

On the contrary assume that,

$A(x) \not\leq B(x)$, $\forall\ x \in X$

$\therefore$ for $x \in X$ such that,

$A(x) > B(x)$

$\therefore \exists\ \alpha \in [0,1]$ such that,

$A(x) > \alpha > B(x)$

$A(x) > \alpha$ & $\alpha > B(x)$

$\Rightarrow A(x) > \alpha$ & $B(x) \ngtr \alpha$

$\Rightarrow x \in \alpha_A^+$ & $x \notin \alpha_B^+$

But $\alpha_A^+ \subseteq \alpha_B^+$

$\therefore x \in \alpha_A^+$ & $x \notin \alpha_B^+$

which is contradiction. $A(x) \leq B(x)$, $\forall\ x \in X$

i.e, $A \subseteq B$

② suppose that $A = B$

$\Leftrightarrow A \subseteq B$ & $B \subseteq A$

$\Leftrightarrow \alpha_A \subseteq \alpha_B$ & $\alpha_B \subseteq \alpha_A$

$\Leftrightarrow \alpha_A = \alpha_B$, $\forall\ \alpha \in [0,1]$

Thus, $A = B$ iff $\alpha_A = \alpha_B$, $\forall\ \alpha \in [0,1]$

Now, suppose that, $A = B$

$\Leftrightarrow A \subseteq B$ & $B \subseteq A$

$\Leftrightarrow \alpha_A^+ \subseteq \alpha_B^+$ & $\alpha_B^+ \subseteq \alpha_A^+$

$\Leftrightarrow \alpha_A^+ = \alpha_B^+$ $\forall\ \alpha \in [0,1]$

Thus, $A = B$ iff $\alpha_A^+ = \alpha_B^+$; $\forall\ \alpha \in [0,1]$

Hence, the proof.

Theorem :-

Statement:

For any $A \in F(x)$ the following property holds.

① $\alpha_A = \bigcap\limits_{\beta < \alpha} \beta_A = \bigcap\limits_{\beta < \alpha} \beta_A^+$

② $\alpha_A^+ = \bigcup\limits_{\alpha < \beta} \beta_A = \bigcup\limits_{\alpha < \beta} \beta_A^+$

Proof :-

① For any $\beta < \alpha$, we have,

$\alpha_A \subseteq \beta_A$

hence, $\alpha_A \subseteq \bigcap\limits_{\beta < \alpha} \beta_A$ ——— ①

Next,

If $x \in \bigcap\limits_{\beta < \alpha} \beta_A$

$\Rightarrow x \in \beta_A$, $\forall\ \beta < \alpha$

Let $\epsilon > 0$ be a ~~the~~ small positive arbitary real number then

$x \in (\alpha-\epsilon)_A$

$\Rightarrow A(x) \geq \alpha-\epsilon$

$\Rightarrow A(x) \geq \alpha$ as $\epsilon \to 0$

$\Rightarrow x \in \alpha_A$

Thus, $x \in \bigcap_{\beta<\alpha} \beta_A \Rightarrow x \in \alpha_A$

$$\bigcap_{\beta<\alpha} \beta_A \subseteq \alpha_A \quad \text{——} (2)$$

from eq (1) & (2)

$$\alpha_A = \bigcap_{\beta<\alpha} \beta_A \quad \text{——} (5)$$

(b) Similarly, $x \in \alpha_A$

$\Rightarrow A(x) \geq \alpha$

If $\beta<\alpha$ then we have,

$A(x) \geq \alpha > \beta$

$\Rightarrow A(x) > \beta \quad \forall\ \beta<\alpha$

$\Rightarrow x \in \beta_A^+$

~~thus~~ $x \in \bigcap_{\beta<\alpha} \beta_A^+$

thus, $x \in \alpha_A \Rightarrow x \in \bigcap_{\beta<\alpha} \beta_A^+$

$$\therefore \alpha_A \subseteq \bigcap_{\beta<\alpha} \beta_A^+ \quad \text{——} (3)$$

next, if $x \in \bigcap_{\beta<\alpha} \beta_A^+$

$\Rightarrow x \in \beta_A^+ \quad \forall\ \beta<\alpha$

$\Rightarrow A(x) > \beta$

let $\epsilon>0$, be small arbitrary real number

then, $\alpha-\epsilon < \alpha$

$\therefore A(x) > \alpha-\epsilon$

$\Rightarrow A(x) \geq \alpha$ as $\epsilon \to 0$

$\Rightarrow x \in \alpha_A$

Thus, $x \in \bigcap_{\beta<\alpha} \beta_A^+ \Rightarrow x \in \alpha_A$

$$\bigcap_{\beta<\alpha} \beta_A^+ \subseteq \alpha_A \quad \text{——} (4)$$

$\therefore$ from (3) & (4)

$$\alpha_A = \bigcap_{\beta<\alpha} \beta_A^+ \quad \text{——} (5)$$

from (5) & (5)

$$\alpha_A = \bigcap_{\beta<\alpha} \beta_A = \bigcap_{\beta<\alpha} \beta_A^+$$

(c) For any $\underset{\alpha<\beta}{\downarrow}$ we have

$x \in \alpha_A^+$

$A(x) > \alpha$

let $\epsilon>0$ be arbitrary real number, then

$\alpha+\epsilon > \alpha$

$\therefore A(x) \geq \alpha+\epsilon > \alpha$

$\Rightarrow A(x) \geq \beta$

$\Rightarrow x \in \beta_A$

$\Rightarrow x \in \bigcup_{\alpha<\beta} \beta_A$

Thus, $x \in \alpha_A^+ \Rightarrow x \in \bigcup_{\alpha<\beta} \beta_A$

$$\therefore \alpha_A^+ \subseteq \bigcup_{\alpha<\beta} \beta_A \quad \text{——} (6)$$

Next, if $x \in \bigcup_{\alpha<\beta} \beta_A$

$\Rightarrow x \in \beta_A$ for some $\alpha<\beta$

$\Rightarrow A(x) \geq \beta > \alpha$

$\Rightarrow A(x) > \alpha$

$\Rightarrow x \in \alpha_A^+$

Thus,

$x \in \bigcup_{\alpha<\beta} \beta_A \Rightarrow x \in \alpha_A^+$

$\therefore \bigcup_{\alpha<\beta} \beta_A \subseteq \alpha_A^+$ ——— (6)

$\therefore$ from eq^n (5) & (6)

$\alpha_A^+ = \bigcup_{\alpha<\beta} \beta_A$ ——— (III)

Similarly,

If $x \in \alpha_A^+$

$\Rightarrow A(x) > \alpha$

let $\epsilon > 0$ be small arbitrary real number then,

$\alpha + \epsilon > \alpha$

then, $A(x) > \alpha + \epsilon > \alpha$

$\Rightarrow A(x) > \beta$

$\Rightarrow x \in \beta_A^+$

$\Rightarrow x \in \bigcup \beta_A^+$

Thus, $x \in \alpha_A^+ \Rightarrow x \in \bigcup_{\alpha<\beta} \beta_A^+$

$\therefore \alpha_A^+ \subseteq \bigcup_{\alpha<\beta} \beta_A^+$ ——— (7)

Next,

$x \in \bigcup_{\alpha<\beta} \beta_A^+ \Rightarrow x \in \alpha^+$

$\Rightarrow x \in \beta_A^+$ for some $\alpha < \beta$

$\Rightarrow A(x) > \beta$

$\Rightarrow A(x) > \beta > \alpha$

$\Rightarrow A(x) > \alpha$

$\Rightarrow x \in \alpha_A^+$

Thus, $x \in \bigcup_{\alpha<\beta} \beta_A^+ \Rightarrow x \in \alpha_A^+$

$\therefore \bigcup_{\alpha<\beta} \beta_A^+ \subseteq \alpha_A^+$ ——— (8)

From (7) & (8)

$\alpha_A^+ = \bigcup_{\alpha<\beta} \beta_A^+$ ——— (IV)

From (III) & (IV)

$\therefore \alpha_A^+ = \bigcup_{\alpha<\beta} \beta_A = \bigcup_{\alpha<\beta} \beta_A^+$

Hence the proof

Theorem :-

If $A_i \in F(X)$, $i \in I$ where "I" is an index set then show that,

(1) $\alpha_{(\bigcup_{i \in I} A_i)} \supseteq \bigcup_{i \in I} \alpha_{A_i}$

(2) $\alpha_{(\bigcap_{i \in I} A_i)} = \bigcap_{i \in I} \alpha_{A_i}$

(3) $\alpha^+_{(\bigcup_{i \in I} A_i)} = \bigcup_{i \in I} \alpha^+_{A_i}$

(4) $\alpha^+_{(\bigcap_{i \in I} A_i)} \supseteq \bigcap_{i \in I} \alpha^+_{A_i}$

Proof :-

let $\{A_i\}_{i \in I}$ be the index family of fuzzy subset of X.

(1) let $x \in \bigcup_{i \in I} \alpha_{A_i}$

$\Rightarrow x \in \alpha_{A_i}$ for some $i \in I$

$\Rightarrow A_i(x) \geq \alpha$

$\Rightarrow \bigvee_{i \in I} A_i(x) \geq \alpha$

$\Rightarrow \bigcup_{i \in I} A_i(x) \geq \alpha$

$\Rightarrow x \in \alpha_{\bigcup_{i \in I} A_i}$

Thus, $x \in \bigcup_{i \in I} \alpha_{A_i} \Rightarrow x \in \alpha_{\bigcup_{i \in I} A_i}$

$\therefore \bigcup_{i \in I} \alpha_{A_i} \subseteq \alpha_{\bigcup_{i \in I} A_i}$

(2) let, $x \in \alpha_{(\bigcap_{i \in I} A_i)}$

$\Leftrightarrow \bigcap_{i \in I} A_i(x) \geq \alpha$

$\Leftrightarrow \bigwedge_{i \in I} A_i(x) \geq \alpha$

$\Leftrightarrow A_i(x) \geq \alpha$

$\Leftrightarrow x \in \alpha_{A_i} \quad \forall\, i \in I$

$\Leftrightarrow x \in \bigcap_{i \in I} \alpha_{A_i}$

thus, $x \in \alpha_{(\bigcap_{i \in I} A_i)} \Leftrightarrow x \in \bigcap_{i \in I} \alpha_{A_i}$

$\therefore \alpha_{\bigcap_{i \in I} A_i} = \bigcap_{i \in I} \alpha_{A_i}$

(3) let, $x \in \alpha^+_{(\bigcup_{i \in I} A_i)}$

$\Leftrightarrow \bigcup_{i \in I} A_i(x) > \alpha$

$\Leftrightarrow \bigvee_{i \in I} A_i(x) > \alpha$

$\Leftrightarrow A_i(x) > \alpha$

$\Leftrightarrow x \in \alpha^+_{A_i}$ for some $i \in I$

$\Leftrightarrow x \in \bigcup_{i \in I} \alpha^+_{A_i}$

thus, $x \in \alpha^+_{(\bigcup_{i \in I} A_i)} \Leftrightarrow x \in \bigcup_{i \in I} \alpha^+_{A_i}$

$\therefore \alpha^+_{(\bigcup_{i \in I} A_i)} = \bigcup_{i \in I} \alpha^+_{A_i}$

(4) let, $x \in \bigcap_{i \in I} \alpha^+_{A_i}$

$\Rightarrow x \in \alpha^+_{A_i} \quad \forall\, i \in I$

$\Rightarrow A_i(x) > \alpha$

$\Rightarrow \bigwedge_{i \in I} A_i(x) > \alpha$

$\Rightarrow \bigcap_{i \in I} A_i(x) > \alpha$

$\Rightarrow x \in \alpha^+_{\bigcap_{i \in I} A_i}$

Thus, $x \in \bigcap_{i \in I} \alpha^+_{A_i} \Rightarrow x \in \alpha^+_{\bigcap_{i \in I} A_i}$

$\therefore \bigcap_{i \in I} \alpha^+_{A_i} \subseteq \alpha^+_{\bigcap_{i \in I} A_i}$

i.e, $\alpha^+_{\bigcap_{i \in I} A_i} \supseteq \bigcap_{i \in I} \alpha^+_{A_i}$

Hence the proof.

* Note:- In the above th^m the equality for (4) in the result $\alpha_{(\bigcup_{i \in I} A_i)} \supseteq \bigcup_{i \in I} \alpha_{A_i}$ need be true.

let $I = N$ & $A_i \in F(x)$, $i \in N$ where $A_i : X \to [0,1]$ defined by where, $A_i : X \to I$, defined by

$A_i(x) = 1 - \frac{1}{i}$, $\forall\ x \in X$

Then, $\bigcup_{i\in N} A_i(x) = \bigvee_{i\in N} A_i(x)$

$= \bigvee_{i\in N}\left(1-\frac{1}{i}\right)$

$= \bigvee_{i\in N}\left(0, \frac{1}{2}, \frac{2}{3}, \dots\right)$

$\bigcup_{i\in N} A_i(x) = 1$

$x \in {}^1\left(\bigcup_{i\in N} A_i\right)$

i.e. $x \in {}^1\left(\bigcup_{i\in N} A_i\right)$ —— ①

Now,

$1_{A_i} = \{x\in X / A_i(x) \geq 1\}$

$= \{x \in X / 1-\frac{1}{i} \geq 1\}$ for $i \in N$

$1_{A_i} = \phi$

$\Rightarrow \bigcup 1_{A_i} = \phi$ —— ②

from eqn ① & ②

$1_{\bigcup_{i\in I} A_i} \supsetneq \bigcup_{i\in I} 1_{A_i}$

hence we have,

$\bigcup_{i\in I} \alpha_{A_i} \subsetneq {}^{\alpha}\left(\bigcup_{i\in I} A_i\right)$

Unit - II

* Special fuzzy set

Defn:

If $A: X\to I$ is the fuzzy set then we define a special fuzzy set denoted by α^A and defined as $\alpha^A(x)$:

$\alpha^A(x) = \alpha$ if $x \in \alpha_A$

$= 0$ if $x \notin \alpha_A$

* Example:

1) let $X = \{x_1, x_2, x_3, x_4, x_5\}$

let $A: X\to I$

defined by $A = \frac{0.2}{x_1} + \frac{0.4}{x_2}+\frac{0.6}{x_3}+\frac{0.8}{x_4}+\frac{1}{x_5}$

then find 0.2^A, 0.4^A, 0.6^A, 0.8^A & 1^A.

$\Rightarrow$ let $A: X\to I$

defined by,

$A = \frac{0.2}{x_1} + \frac{0.4}{x_2}+\frac{0.6}{x_3}+\frac{0.8}{x_4}+\frac{1}{x_5}$

then,

$0.2_A = \{x\in X / A(x) \geq 0.2\}$

$= \{x_1, x_2, x_3, x_4, x_5\}$

$0.4_A = \{x\in X / A(x) \geq 0.4\}$

$= \{x_2, x_3, x_4, x_5\}$

$0.6_A = \{x\in X / A(x) \geq 0.6\}$

$= \{x_3, x_4, x_5\}$

$0.8_A = \{x\in X / A(x) \geq 0.8\}$

$= \{x_4, x_5\}$

$1_A = \{x\in X / A(x) \geq 1\}$

$= \{x_5\}$

then special fuzzy sets

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$0.2^A(x) = 0.2$ if $x \in 0.2_A$
$= 0$ if $x \notin 0.2_A$

$0.2^A = \left\{ \frac{0.2}{x_1} + \frac{0.2}{x_2} + \frac{0.2}{x_3} + \frac{0.2}{x_4} + \frac{0.2}{x_5} \right\}$

$0.4^A = \left\{ \frac{0.4}{x_1} + \frac{0.4}{x_2} + \frac{0.4}{x_3} + \frac{0.4}{x_4} + \frac{0.4}{x_5} \right\}$

$0.6^A = \left\{ \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5} \right\}$

$0.8^A = \left\{ \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0.8}{x_4} + \frac{0.8}{x_5} \right\}$

$1^A = \left\{ \frac{0}{x_1} + \frac{0}{x_2} + \frac{0}{x_3} + \frac{0}{x_4} + \frac{1}{x_5} \right\}$

* Note :-

1) Consider the union of all special fuzzy set.
i.e. $0.2^A \cup 0.4^A \cup 0.6^A \cup 0.8^A \cup 1^A = \left\{ \frac{0.2}{x_1} + \frac{0.4}{x_2} + \frac{0.6}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5} \right\}$
$= A$

Thus,
$0.2^A \cup 0.4^A \cup 0.6^A \cup 0.8^A \cup 1^A = A$
Thus for any fuzzy set A
$$A = \bigcup_{\alpha \in [0,1]} \alpha^A$$

2) $\alpha^A = \alpha(\alpha_{A(x)})$
where, $\alpha_{A(x)}$ is the characteristic function of α_A
e.g. $\alpha = 0.6$
$0.6^A = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5}$

$0.6_A = \{x_3, x_4, x_5\}$

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$0.6_{A(x)} = \frac{0}{x_1} + \frac{0}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} + \frac{1}{x_5}$

$0.6\left(0.6_{A(x)}\right) = \frac{0}{x_1} + \frac{0}{x_2} + \frac{0.6}{x_3} + \frac{0.6}{x_4} + \frac{0.6}{x_5}$

Thus, $0.6^A = 0.6\,(0.6_{A(x)})$
i.e. $\alpha^A = \alpha\,(\alpha_{A(x)})$

* First Decomposition Theorem :-

Statement :-
For every $A \in F(x)$, $A = \bigcup_{\alpha \in [0,1]} \alpha^A$

where, α^A is a special fuzzy set

proof :-
let $x \in X$ be arbitrary.
and let $A(x) = a$
where $a \in [0,1]$
consider, $\left(\bigcup_{\alpha \in [0,1]} \alpha^A\right)(x) = \bigvee_{\alpha \in [0,1]} \{\alpha^A(x)\}$

$$\bigcup_{\alpha \in [0,1]} \alpha^A(x) = \left(\bigvee_{\alpha \in [0,a]} \{\alpha^A(x)\}\right) \vee \left(\bigvee_{\alpha \in (a,1]} \{\alpha^A(x)\}\right) \quad \text{—(1)}$$

Now, if $\alpha \in [0,a]$
$\Rightarrow \alpha \leq a$
$\Rightarrow a \geq \alpha$
but $a = A(x)$
$\Rightarrow A(x) \geq \alpha$
$\Rightarrow x \in \alpha_A$

$\Rightarrow \alpha^A(x) = \alpha$

$\therefore \bigvee_{\alpha \in [0,a]} \{\alpha^A(x)\} = \bigvee_{\alpha \in [0,a]} \alpha = a$

$\Rightarrow \bigvee_{\alpha \in [0,a]} \{\alpha^A(x)\} = a$

$$\Rightarrow \bigvee_{\alpha \in [0,a]} \{\alpha^A(x)\} = A(x)$$

if $\alpha \in (a,1]$

$\Rightarrow \alpha > a$

$\Rightarrow a < \alpha$

$\Rightarrow A(x) < \alpha$

$\Rightarrow A(x) \not\geq \alpha$

$\Rightarrow x \notin \alpha_A$

$\Rightarrow \alpha^A(x) = 0$

$\therefore \bigvee_{\alpha \in [a,1]} \{\alpha^A(x)\} = \bigvee_{\alpha \in (a,1]} 0 = 0$

$\therefore \bigvee_{\alpha \in (a,1]} \{\alpha^A(x)\} = 0$

$\therefore$ eqn ② becomes

$$\left(\bigcup_{\alpha \in [0,1]} \alpha^A\right)(x) = \cancel{A(x) \vee 0} = a \vee 0 = a$$

$$\bigcup_{\alpha \in [0,1]} \alpha^A(x) = A(x)$$

Thus, $\bigcup_{\alpha \in [0,1]} \alpha^A = A$

Hence the proof.

* Second decomposition theorem :-

Statement :- For any $A \in F(x)$, $A = \bigcup_{\alpha \in [0,1]} \alpha^{+A}$

where, $\alpha^{+A}(x) = \alpha$, if $x \in \alpha_A^+$
$= 0$, if $x \notin \alpha_A^+$

State & prove

5 marks

proof :-

let $x \in X$ be arbitary.

let $A(x) = a$

where, $a \in [0,1]$

Consider, $\left(\bigcup_{\alpha \in [0,1]} \alpha^{+A}\right)(x) = \bigvee_{\alpha \in [0,1]} \alpha^{+A}(x)$

$$= \left(\bigvee_{\alpha \in [0,a)} \alpha^{+A}(x)\right) \vee \left(\bigvee_{\alpha \in [a,1]} \alpha^{+A}(x)\right) \quad \text{— ①}$$

Now, if $\alpha \in [0,a)$

$\Rightarrow \alpha < a$

$\Rightarrow \alpha < A(x)$

$\Rightarrow A(x) > \alpha$

$\Rightarrow x \in \alpha_A^+$

$\Rightarrow \alpha_A^{+A}(x) = \alpha$

hence,

$$\bigvee_{\alpha \in [0,a)} \alpha^{+A}(x) = \bigvee_{\alpha \in [0,a)} \alpha = a$$

$$\bigvee_{\alpha \in [0,a)} \alpha^{+A}(x) = a$$

& if $\alpha \in [a,1]$

$\Rightarrow a \leq \alpha$

$\Rightarrow A(x) \leq \alpha$

$\Rightarrow A(x) \not> \alpha$

$\Rightarrow x \notin \alpha_A^+$

$\Rightarrow \alpha^{+A}(x) = 0$

$\therefore \bigvee_{\alpha \in [a,1]} \alpha^{+A}(x) = \bigvee_{\alpha \in [a,1]} 0 = 0$

$\therefore \bigvee_{\alpha \in [a,1]} \alpha^{+A}(x) = 0$

eqⁿ ① becomes,

$$\left(\bigcup_{\alpha\in[0,1]} {}^{\alpha+}A\right)(x) = a \vee 0$$
$$= a$$
$$\left(\bigcup_{\alpha\in[0,1]} {}^{\alpha+}A\right)(x) = A(x)$$

Thus, $\bigcup_{\alpha\in[0,1]} {}^{\alpha+}A = A$

Hence the proof.

* Example.

1) let A & B be the fuzzy sets define on the universal set Z, whose membership function is given by,

$$A = \frac{0.5}{-1} + \frac{1}{0} + \frac{0.5}{1} + \frac{0.3}{2} \quad \&$$

$$B = \frac{0.5}{2} + \frac{0.2}{3} + \frac{0.5}{4} + \frac{0.3}{5}$$

let F be a Function defined by

① $f(x_1, x_2) = x_1 \cdot x_2$, find $F(A,B)$

② $f(x_1, x_2) = x_1 + x_2$, find. $f(A,B)$

→ let $A : Z \to I$ &

$B : Z \to I$

then $f(A,B) : Z \to I$, given by

$$f(A,B)(z) = \bigvee_{(u,v)\in f^{-1}(z)} (A,B)(u,v)$$

$$f(A,B)(z) = \bigvee_{f(u,v)\in z} (A(u) \wedge B(v))$$

support of A = {-1, 0, 1, 2}

support of B = {2, 3, 4, 5}

① let f be a function defined by,

$f(x_1, x_2) = x_1 \cdot x_2$

∴ support of f(A,B) = {-5, -4, -3, -2, 0, 2, 3, 4, 5, 6, 8, 10}

then we have,

$$f(A,B)(-5) = \bigvee_{-5\in z} (A(-1) \wedge B(5))$$
$$= \bigvee_{-5\in z} [\min(A(-1), B(5))]$$
$$= \bigvee_{-5\in z} \min(0.5, 0.3)$$

$$f(A,B)(-5) = 0.3$$

$$f(A,B)(-4) = \bigvee_{-4\in z} (A(-1) \wedge B(4))$$
$$= \bigvee_{-4\in z} [\min(A(-1), B(4))]$$
$$= \bigvee_{-4\in z} \min(0.5, 0.5)$$

$$f(A,B)(-4) = 0.5$$

$$f(A,B)(-3) = \bigvee_{-3\in z} (A(-1) \wedge B(3))$$
$$= \bigvee_{-3\in z} \min(A(-1), B(3))$$
$$= \bigvee_{-3\in z} \min(0.5, 0.2)$$
$$= 0.2$$

$$f(A,B)(-2) = \bigvee_{-2 \in Z} (A(-1) \wedge B(2))$$

$$= \bigvee_{-2 \in Z} \min [A(-1), B(2)]$$

$$= \bigvee_{-2 \in Z} \min (0.5, 0.3)$$

$$f(A,B)(-2) = 0.3$$

$$f(A,B)(4) = \bigvee_{4 \in Z} \begin{pmatrix} \min(A(1), B(4)), \\ \min(A(2), B(2)) \end{pmatrix}$$

$$= \bigvee_{4 \in Z} (\min(0.5, 0.5), \min(0.3, 0.3))$$

$$= \bigvee_{4 \in Z} (0.5, 0.3)$$

$$f(A,B)(4) = 0.5$$

$$f(A,B)(0) = \bigvee_{0 \in Z} \begin{pmatrix} \min(A(0), B(2)), \min(A(0), B(3)), \\ \min(A(0), B(4)), \min(A(0), B(5)) \end{pmatrix}$$

$$= \bigvee_{0 \in Z} \begin{pmatrix} \min(1, 0.3), \min(1, 0.2), \\ \min(1, 0.5), \min(1, 0.3) \end{pmatrix}$$

$$= \bigvee_{0 \in Z} (0.3, 0.2, 0.5, 0.3)$$

$$f(A,B)(0) = 0.5$$

$$f(A,B)(2) = \bigvee_{2 \in Z} (A(1) \wedge B(2))$$

$$= \bigvee_{2 \in Z} \min (A(1), B(2))$$

$$= \bigvee_{2 \in Z} \min (0.5, 0.3)$$

$$= 0.3$$

$$f(A,B)(3) = \bigvee_{3 \in Z} (A(1) \wedge B(3))$$

$$= \bigvee_{3 \in Z} \min (0.5, 0.2)$$

$$f(A,B)(3) = 0.2$$

$$\cancel{f(A,B)(4)} = \cancel{\bigvee_{4 \in Z} (A(1) \wedge B(4))}$$

$$= \cancel{\bigvee_{4 \in Z} \min (A(1), B(4))}$$

$$= \cancel{\bigvee_{4 \in Z} \min (0.5, 0.5)}$$

$$\cancel{f(A,B)(4)} = \cancel{0.5}$$

$$f(A,B)(5) = \bigvee_{4 \in Z} (A(1) \wedge B(5))$$

$$= \bigvee_{4 \in Z} \min (0.5, 0.3)$$

$$f(A,B)(5) = 0.3$$

$$f(A,B)(6) = \bigvee_{6 \in Z} \min (A(2), B(3))$$

$$= \bigvee_{6 \in Z} \min (0.3, 0.2)$$

$$f(A,B)(6) = 0.2$$

$$f(A,B)(8) = \bigvee_{8 \in Z} \min (A(2), B(4))$$

$$= \bigvee_{8 \in Z} \min (0.3, 0.5)$$

$$f(A,B)(8) = 0.3$$

$$f(A,B)(10) = \bigvee_{10 \in Z} \min (A(2), B(5))$$

$$= \bigvee_{10 \in Z} \min (0.3, 0.3)$$

$$f(A,B)(10) = 0.3$$

$$F(A,B) = \frac{0.3}{-5} + \frac{0.5}{-4} + \frac{0.2}{-3} + \frac{0.3}{-2} + \frac{0.5}{0} + \frac{0.3}{2} + \frac{0.2}{3}$$
$$+ \frac{0.5}{4} + \frac{0.5}{5} + \frac{0.3}{6} + \frac{0.3}{8} + \frac{0.3}{10}$$

② let F be a function defined by,

$F(x_1, x_2) = x_1 + x_2$

support of $F(A,B) = \{1, 2, 3, 4, 5, 6, 7\}$

then we have,

$F(A,B)(1) = \bigvee_{1\in z} (A(-1) \wedge B(2))$

$= \bigvee_{1\in z} \min(A(-1), B(2))$

$= \bigvee_{1\in z} \min(0.5, 0.3)$

$= 0.3$

$F(A,B)(2) = \bigvee_{2\in z} [A(0) \wedge B(2), A(-1) \wedge A(3),]$

$= \bigvee_{2\in z} \min[(A(0), B(2))], \min[A(-1), B(3)]$

$= \bigvee_{2\in z} \min[(1, 0.3), \min(0.5, 0.2)]$

$= \bigvee_{2\in z} (0.3), (0.2)$

$= 0.3$

$F(A,B)(3) = \bigvee_{3\in z} [A(-1) \wedge B(4), A(0) \wedge B(3), A(1) \wedge B(2),$

$= \bigvee_{3\in z} \min(A(-1), B(4)), \min(A(0), B(3))$
$\min(A(1), B(2))$

$= \bigvee_{3\in z} \min(0.5, 0.5), \min(1, 0.2)$
$\min(0.5, 0.3)$

$= \bigvee_{3\in z} (0.5, 0.2, 0.3)$

$= 0.5$

$F(A,B)(4) = \bigvee_{4\in z} [A(-1) \wedge B(5), A(0) \wedge B(4), A(1) \wedge B(3), A(2) \wedge B(2)]$

$= \bigvee_{4\in z} [\min(A(-1), B(5)), \min(A(0), B(4)), \min(A(1), B(3)), \min A(2), B(2)]$

$= \bigvee_{4\in z} [\min(0.5, 0.3), \min(1, 0.5), \min(0.5, 0.2), \min(0.3, 0.3)]$

$= \bigvee_{4\in z} [0.3, 0.5, 0.2, 0.3]$

$= 0.5$

$F(A,B)(5) = \bigvee_{5\in z} [A(1) \wedge B(4), A(2) \wedge B(5), B(0) \wedge B(5)]$

$= \bigvee_{5\in z} [\min(A(0), B(5)), \min(A(1), B(4)), \min(A(2), B(3))]$

$= \bigvee_{5\in z} [\min(1, 0.3), \min(0.5, 0.5), \min(0.3, 0.2)]$

$= \bigvee_{5\in z} (0.3, 0.5, 0.2)$

$= 0.5$

$F(A,B)(6) = \bigvee_{6\in z} [A(1) \wedge B(5), A(2) \wedge B(4)]$

$= \bigvee_{6\in z} [\min(A(1), B(5)), \min(A(2), B(4))]$

$= \bigvee_{6\in z} [\min(0.5, 0.3), \min(0.3, 0.5)]$

$= \bigvee_{6\in z} [0.3, 0.3]$ ③

$= 0.3$

$$F(A,B)(7) = \bigvee_{7 \in Z} [A(2) \wedge B(5)]$$
$$= \bigvee_{7 \in Z} \min(A(2), B(5))$$
$$= \bigvee_{7 \in Z} \min(0.3, 0.3)$$
$$= 0.3$$

$$F(A,B) = \frac{0.3}{1} + \frac{0.3}{2} + \frac{0.5}{3} + \frac{0.5}{4} + \frac{0.5}{5} + \frac{0.3}{6} + \frac{0.3}{7}$$

* Extension Principle :-

Def^n:-

let $f: X \to Y$ be a function define,

$f: \mathcal{F}(x) \to \mathcal{F}(y)$ as,

$$\boxed{(f(A))(y) = \sup_{x \in f^{-1}(y)} A(x)}$$
$$= \sup_{y = f(x)} A(x)$$

& define,

$f^{-1}: \mathcal{F}(y) \to \mathcal{F}(x)$ as,

$$\boxed{(f^{-1}(B))(x) = B(f(x))}$$

where, $B \in \mathcal{F}(y)$

* Theorem :-

If $f: X \to Y$ be a crisp function for $A_i \in \mathcal{F}(x)$ & $B_i \in \mathcal{F}(y)$, $i \in I$ then following properties holds true.

① $f(A) = \phi$ iff $A = \phi$

② $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

③ $f(\bigcup_i A_i) = \bigcup_i f(A_i)$

④ $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$

⑤ $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

⑥ $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$

⑦ $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$

⑧ $\overline{f^{-1}(B)} = f^{-1}(\bar{B})$

⑨ $A \subseteq f^{-1}(f(A))$

⑩ $B = f(f^{-1}(B))$

proof :-

① let $A = \phi$

then $A(x) = 0 \quad \forall x \in X$

$$\therefore f(A)(y) = \bigvee_{x \in f^{-1}(y)} A(x)$$
$$= \bigvee_{x \in f^{-1}(y)} 0$$
$$= 0$$

Thus, $f(A)(y) = 0 \quad \forall y \in Y$

$\Rightarrow f(A) = \phi$

Conversely,

if $f(A) = \phi$ then

$f(A)(y) = 0, \quad \forall y \in Y$

$\Rightarrow \bigvee_{x \in f^{-1}(y)} A(x) = 0$

$\Rightarrow A(x) = 0 \quad \forall x \in f^{-1}(y)$

$\Rightarrow A(x) = 0 \quad \forall x \in X$

$\Rightarrow A = \phi$

Thus, $f(A) = \phi$ iff $A = \phi$

③ let $A_1 \subseteq A_2$

we have,

$A_1(x) \leq A_2(x), \quad \forall x \in X$

$\bigvee_{x \in f^{-1}(y)} A_1(x) \leq \bigvee_{x \in f^{-1}(y)} A_2(x)$

$\Rightarrow f(A_1)(y) \leq f(A_2)(y) \quad \forall y \in Y$

$\Rightarrow f(A_1) \subseteq f(A_2)$

Thus, $A_1 \subseteq A_2$

$\Rightarrow f(A_1) \subseteq f(A_2)$

④ For $y \in Y$

consider, $f\left(\bigcup_i A_i\right)(y)$

$= \bigvee_{x \in f^{-1}(y)} \left(\bigcup_i A_i\right)(x)$

$= \bigvee_{x \in f^{-1}(y)} \left(\bigvee_i A_i\right)(x)$

$= \bigvee_i \left(\bigvee_{x \in f^{-1}(y)} A_i(x)\right)$

$= \bigvee_i f(A_i)(y)$

$= \bigcup_i f(A_i)(y)$

Thus, $f\left(\bigcup_i A_i\right)(y) = \bigcup_i f(A_i)(y)$

i.e, $f\left(\bigcup_i A_i\right) = \bigcup_i f(A_i)$

④ For $y \in Y$,

consider $f\left(\bigcap_i A_i\right)(y)$

$= \bigvee_{x \in f^{-1}(y)} \left(\bigcap_i A_i\right)(x)$

$f^{-1}B(x) = B(f(x))$

$= \bigvee_{x \in f^{-1}(y)} \left(\bigwedge_i A_i\right)(x)$

$\leq \bigwedge_i \left(\bigvee_{x \in f^{-1}(y)} A_i(x)\right)$

$\leq \bigwedge_i f(A_i)(y)$

$\leq \bigcap_i f(A_i)(y)$

Thus, $f\left(\bigcap_i A_i\right)(y) \leq \bigcap_i f(A_i)(y)$

$\Rightarrow f\left(\bigcap_i A_i\right) \subseteq \bigcap_i f(A_i)$

⑤ If $B_1 \subseteq B_2$

then, $B_1(y) \leq B_2(y) \quad \forall y \in Y$

consider,

$f^{-1}B_1(x) = B_1(f(x)) \quad \forall x \in X$

$\leq B_2(f(x))$

$= f^{-1}(B_2(x))$

Thus, $f^{-1}(B_1(x)) \leq f^{-1}(B_2(x)) \quad \forall x \in X$

i.e., $f^{-1}(B_1) \subseteq f^{-1}(B_2)$

⑥ ~~let~~.

consider, $f^{-1}\left(\bigcup_i B_i\right)(x)$

$= \bigcup_i B_i f(x)$

$= \bigvee_i B_i f(x)$

$= \bigvee_i f^{-1} B_i(x)$

$= \bigcup_i f^{-1} B_i(x)$

Thus, $f^{-1}(\bigcup_i B_i)(x) = (\bigcup_i f^{-1}(B_i))(x)$

i.e, $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$

⑦ consider, $f^{-1}(\bigcap_i B_i)(x)$

$= \bigcap_i B_i f(x)$

$= \bigwedge_i B_i f(x)$

$= \bigwedge_i f^{-1} B_i(x)$

$= \bigcap_i f^{-1} B_i(x)$

Thus, $f^{-1}(\bigcap_i B_i)(x) = (\bigcap_i f^{-1}(B_i))(x)$

i.e, $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$

⑧ For any $x \in X$

$\overline{f^{-1}(B)}(x) = 1 - f^{-1}(B)(x)$

$= 1 - \cancel{B(x)}\ B(f(x))$

$= \overline{B} f(x)$

$= f^{-1}(\overline{B}(x))$

Thus, $\overline{f^{-1}(B(x))} = f^{-1}(\overline{B}(x))$

i.e, $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$

⑨ consider, $f^{-1}(f(A))(x)$

$= f(A) f(x)$

$= \cancel{\sup\ A(x)}$

$= \bigvee_{z \in f^{-1}(f(x))} A(z)$

clearly, $x \in f^{-1}(f(x))$

hence, $f^{-1}(f(A))(x) \geq A(x) \quad \forall x \in X$

i.e, $A(x) \leq f^{-1}(f(A))(x)$

$A \subseteq f^{-1}(f(A))$

⑩ For any $y \in Y$,

consider, $f(f^{-1}(B))(y)$

$= \bigvee_{x \in f^{-1}(y)} f^{-1}(B)(x)$

$= \bigvee_{x \in f^{-1}(y)} B(f(x))$

$= \bigvee_{f(x) \in y} B(f(x))$

$= \bigvee_{f(x) = y \in Y} B(f(x))$

$= \bigvee_{y \in Y} B(f(y))$

$= B(y)$

Thus, $f(f^{-1}(B))(y) = B(y)$

i.e, $f(f^{-1}(B)) = B$

$\Rightarrow B = f(f^{-1}(B))$

Hence the proof.

1) ✱ Example:-

let the universal set is given by

$X = \{0, 1, 2, 3, \ldots, 10\}$

let $f : X \to Y$ be a function defined by

$f(x) = x^2$

where, $Y = \{0, 1, 4, 9, 16, \ldots, 100\}$

If D is a fuzzy set, defined on Y by

$D = \frac{0.5}{4} + \frac{0.6}{16} + \frac{0.7}{25} + \frac{1}{100}$

find $f^{-1}(D)$

⇒ Given that,

$f: X \longrightarrow Y$

be defined by

$f(x) = x^2$

where, $X = \{0, 1, 2, 3, \dots, 10\}$

& $Y = \{0, 4, 9, 16, \dots 100\}$

let D is fuzzy set defined on Y as

$D = \frac{0.5}{4} + \frac{0.6}{16} + \frac{0.7}{25} + \frac{1}{100}$

∴ $f^{-1}(D)$ is a fuzzy set defined on X

i.e. $f^{-1}(D): X \longrightarrow I$ by

$f^{-1}(D)(x) = D(f(x)) \quad \forall x \in X$

then $f^{-1}(D)(0) = D(f(0)) = D(0) = 0$

$f^{-1}(D)(1) = D(f(1)) = D(1^2) = D(1) = 0$

$f^{-1}(D)(2) = D(f(2)) = D(2^2) = D(4) = 0.5$

$f^{-1}(D)(3) = D(f(3)) = D(3^2) = D(9) = 0$

$f^{-1}(D)(4) = D(f(4)) = D(4^2) = D(16) = 0.6$

$f^{-1}(D)(5) = D(f(5)) = D(5^2) = D(25) = 0.7$

$f^{-1}(D)(6) = D(f(6)) = D(6^2) = D(36) = 0$

$f^{-1}(D)(7) = D(f(7)) = D(7^2) = D(49) = 0$

$f^{-1}(D)(8) = D(f(8)) = D(8^2) = D(64) = 0$

$f^{-1}(D)(9) = D(f(9)) = D(9^2) = D(81) = 0$

$f^{-1}(D)(10) = D(f(10)) = D(10^2) = D(100) = 1$

Thus $f^{-1}(D) = \frac{0}{0} + \frac{0}{1} + \frac{0.5}{2} + \frac{0}{3} + \frac{0.6}{4} + \frac{0.7}{5} + \frac{0}{6} + \frac{0}{7} + \frac{0}{8} + \frac{0}{9} + \frac{1}{10}$

2) let $F: X \to Y$ be defined by $f(x) = x + 1$

where, $X = \{0, 1, 2, 3, \dots, 10\}$ & $Y = \{1, 2, 3, 4, \dots, 11\}$

(i) let D is a fuzzy set defined on Y &

$D = \frac{0.5}{3} + \frac{0.6}{6} + \frac{0.7}{7} + \frac{1}{10}$

Find $f^{-1}(D)$.

(ii) let A is a fuzzy set defined on Y as

$A = \frac{0.3}{2} + \frac{0.4}{4} + \frac{0.7}{6} + \frac{0.8}{7} + \frac{1}{10} + \frac{0.2}{11}$ find $f^{-1}(A)$

⇒ (i) Given that, $F: X \longrightarrow Y$

be defined by $f(x) = x + 1$

where, $X = \{0, 1, 2, 3, \dots, 10\}$ & $Y = \{1, 2, 3, 4, \dots, 11\}$

let D is a fuzzy set defined on Y as

$D = \frac{0.5}{3} + \frac{0.6}{6} + \frac{0.7}{7} + \frac{1}{10}$

∴ $f^{-1}(D)$ is a fuzzy set defined on X.

i.e. $f^{-1}(D): X \to I$ by

$f^{-1}(D)(x) = D(f(x)) \quad \forall x \in X$

then, $f^{-1}(D)(0) = D(f(0)) = D(1) = 0$

$f^{-1}(D)(1) = D(f(1)) = D(2) = 0$

$f^{-1}(D)(2) = D(f(2)) = D(3) = 0.5$

$f^{-1}(D)(3) = D(f(3)) = D(3+1) = D(4) = 0$

$f^{-1}(D)(4) = D(f(4)) = D(5) = 0$

$f^{-1}(D)(5) = D(f(5)) = D(6) = 0.6$

$f^{-1}(D)(6) = D(f(6)) = D(7) = 0.7$

$f^{-1}(D)(7) = D(f(7)) = D(8) = 0$

$f^{-1}(D)(8) = D(f(8)) = D(9) = 0$

$f^{-1}(D)(9) = D(f(9)) = D(10) = 1$

$f^{-1}(D)(10) = D(f(10)) = D(11) = 0$

Thus, $f^{-1}(D) = \frac{0}{0} + \frac{0}{1} + \frac{0.5}{2} + \frac{0}{3} + \frac{0}{4} + \frac{0.6}{5} + \frac{0.7}{6} + \frac{0}{7} + \frac{0}{8} + \frac{1}{9} + \frac{0}{10}$

(ii) Given that, $f: X \to Y$
be defined by, $f(x) = x+1$
where, $X = \{0, 1, 2, 3, \ldots, 10\}$ &
$Y = \{1, 2, 3, 4, \ldots, 11\}$
let, A be fuzzy set defined by Y as

$$A = \frac{0.3}{2} + \frac{0.4}{4} + \frac{0.7}{6} + \frac{0.8}{7} + \frac{1}{10} + \frac{0.2}{11}$$

$\therefore f^{-1}(A)$ is fuzzy set defined by X
i.e, $f^{-1}(A): X \to I$ by
$f^{-1}(A)(x) = Af(x) \quad \forall x \in X.$

then,
$f^{-1}(A)(0) = Af(0) = A(0+1) = A(1) = 0$
$f^{-1}(A)(1) = Af(1) = A(2) = 0.3$
$f^{-1}(A)(2) = Af(2) = A(3) = 0$
$f^{-1}(A)(3) = Af(3) = A(4) = 0.4$
$f^{-1}(A)(4) = Af(4) = A(5) = 0$
$f^{-1}(A)(5) = Af(5) = A(6) = 0.7$
$f^{-1}(A)(6) = Af(6) = A(7) = 0.8$
$f^{-1}(A)(7) = Af(7) = A(8) = 0$
$f^{-1}(A)(8) = Af(8) = A(9) = 0$
$f^{-1}(A)(9) = Af(9) = A(10) = 1$
$f^{-1}(A)(10) = Af(10) = A(11) = 0.2$

Thus, $f^{-1}(A) = \frac{0}{0} + \frac{0.3}{1} + \frac{0}{2} + \frac{0.4}{3} + \frac{0}{4} + \frac{0.7}{5} + \frac{0.8}{6} + \frac{0}{7} + \frac{0}{8} + \frac{1}{9} + \frac{0.2}{10}$

* Theorem :-
Statement: let $f: X \to Y$ be a function then
for any $A \in \mathcal{F}(X)$ & for all $\alpha \in [0,1]$

(i) $\alpha^{+}_{[f(A)]} = f(\alpha^{+}_{A})$

(ii) $\alpha_{[f(A)]} = f(\alpha_{A})$

Proof:

$\Rightarrow$ (i) let $y \in Y$
let $y \in \alpha^{+}_{[f(A)]}$

$\Rightarrow f(A)(y) > \alpha$

$\Rightarrow \bigvee_{x \in f^{-1}(y)} A(x) > \alpha$

$\Rightarrow A(x) > \alpha$ for some $x \in f^{-1}(y)$

$\Rightarrow x \in \alpha^{+}_{A}$ for some $x \in f^{-1}(y)$
i.e, $f(x) = y$

$\Rightarrow f(x) \in f(\alpha^{+}_{A})$

$\Rightarrow y \in f(\alpha^{+}_{A})$

Thus, $y \in \alpha^{+}_{[f(A)]} \Rightarrow y \in f(\alpha^{+}_{A})$

i.e, $\alpha^{+}_{[f(A)]} \subseteq f(\alpha^{+}_{A})$ —— (1)

Conversly,
let, $y \in f(\alpha^{+}_{A})$
$\exists\ x \in \alpha^{+}_{A}$ s.t. $y = f(x)$
$\Rightarrow A(x) > \alpha$ for some x s.t. $x = f^{-1}(y)$
$\Rightarrow \bigvee_{x \in f^{-1}(y)} A(x) > \alpha$

$\Rightarrow f(A)(y) > \alpha$
$\Rightarrow y \in \alpha^{+}_{[f(A)]}$

Thus, $y \in f(\alpha^+_A) \Rightarrow y \in \alpha^+_{[f(A)]}$

i.e, $f(\alpha^+_A) \subseteq \alpha^+_{[f(A)]}$ ——— ②

from ① & ②

$\alpha^+_{[f(A)]} = f(\alpha^+_A)$

(ii) let. $y \in f(\alpha_A)$

$\Rightarrow \exists\ x \in \alpha_A$ s.t. $y = f(x)$

$\Rightarrow A(x) \geq \alpha$ for some x, s.t. $x = f^{-1}(y)$

$\Rightarrow \bigvee_{x \in f^{-1}(y)} A(x) \geq \alpha$

$\Rightarrow f(A)(y) \geq \alpha$

$\Rightarrow y \in \alpha_{[f(A)]}$

Thus, $y \in f(\alpha_A) \Rightarrow y \in \alpha_{[f(A)]}$

i.e, $f(\alpha_A) \subseteq \alpha_{[f(A)]}$

i.e, $\alpha_{[f(A)]} \supseteq f(\alpha_A)$

Theorem :-

let $f: X \to Y$ be a function then for $A \in F(X)$

$f(A) = \bigcup_{\alpha \in [0,1]} f(\alpha^{+A})$ where α^{+A} is a special fuzzy set defined by,

$\alpha^{+A} = \alpha$ if $x \in \alpha^+_A$

$= 0$ if $x \notin \alpha^+_A$

proof:-

by decomposition thm, we have,

$A = \bigcup_{\alpha \in [0,1]} \alpha^{+A}$

Then, $f(A) = f\left(\bigcup_{\alpha \in [0,1]} \alpha^{+A}\right)$

$\Rightarrow f(A)(y) = f\left(\bigcup_{\alpha \in [0,1]} \alpha^{+A}\right)(y)$

$= \bigvee_{x \in f^{-1}(y)} \left(\bigcup_{\alpha \in [0,1]} \alpha^{+A}\right)(x)$

$= \bigvee_{x \in f^{-1}(y)} \left(\bigvee_{\alpha \in [0,1]} \alpha^{+A}\right)(x)$

$= \bigvee_{\alpha \in [0,1]} \left(\bigvee_{x \in f^{-1}(y)} \alpha^{+A}(x)\right)$

$= \bigvee_{\alpha \in [0,1]} f(\alpha^{+A}(y))$

Thus, $f(A)(y) = \bigcup_{\alpha \in [0,1]} f(\alpha^{+A})(y)$

i.e, $f(A) = \bigcup_{\alpha \in [0,1]} f(\alpha^{+A})$

* Unit - 3 *

* Operation on fuzzy set:

General fuzzy compliment operators

A function $c:[0,1] \rightarrow [0,1]$ defined by $\boxed{c(x)=1-x}$ is called a standard fuzzy compliment.

For this fuzzy compliments
$c(0)=1$ & $c(1)=0$ and $x \leq y \Rightarrow c(x) \geq c(y)$

$\therefore$ we define a general fuzzy compliment operator as fuzzy compliment.

A function $c:[0,1] \rightarrow [0,1]$ which satisfies

<c_1> $c(0)=1$, $c(1)=0$

<c_2> $a \leq b \Rightarrow c(a) \geq c(b)$

is called a fuzzy compliment

In addition to if,

<c_3> c is continuous &

<c_4> c is ~~involutive~~ involutive

i.e., $\boxed{c(c(a))=a}$ $\forall a \in [0,1]$

then c is called a continuous & involutive fuzzy compliment.

1) Ex) Show that, standard fuzzy compliment defined by, $c(x)=1-x$ is continuous & involutive fuzzy compliment.

$\Rightarrow$ c is function defined by

$c:[0,1] \rightarrow [0,1]$

$c(x) = 1-x$

be a standard fuzzy compliment.

then,

<c_1> $c(0)=1-0=1$ & $c(1)=1-1=0$

<c_2> let $a \leq b$

$\Rightarrow -a \geq -b$

$\Rightarrow 1-a \geq 1-b$

$\Rightarrow c(a) \geq c(b)$

$\therefore$ c is a fuzzy compliment.

<c_3> $c(a)=1-a$

clearly, c is continuous

<c_4> $c(c(a)) = c(1-a)$

$= 1-(1-a)$

$= 1-1+a$

$c(c(a)) = a$

hence c is continuous involutive fuzzy compliment.

2) A function $c:[0,1] \rightarrow [0,1]$ is defined by $c(a) = \frac{1}{2}(1+\cos \pi a)$ show that c is continuous fuzzy compliment but not involutive

$\Rightarrow$ let $c: [0,1] \rightarrow [0,1]$ be a function defined by

$c(a) = \frac{1}{2}(1+\cos \pi a)$

$= \frac{1}{2}\left(2\cos^2\left(\frac{\pi a}{2}\right)\right)$

$c(a) = \cos^2\left(\frac{\pi a}{2}\right)$

<c_1> $c(0) = \cos^2\left(\frac{\pi(0)}{2}\right) = \cos^2(0) = (1)^2 = 1$

&

$c(1) = \cos^2\left(\frac{\pi(1)}{2}\right) = \cos^2\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right)^2 = 0^2 = 0$

<c_2> let $a \leq b$

$\Rightarrow \pi a \leq \pi b$

$1 \geq 0$

$\Rightarrow \cos \pi a \geq \cos \pi b$

$\Rightarrow 1+\cos \pi a \geq 1+\cos \pi b$

$\Rightarrow \frac{1}{2}(1+\cos \pi a) \geq \frac{1}{2}(1+\cos \pi b)$

$\Rightarrow \cancel{c(a) \geq c(b)}$ $c(a) \geq c(b)$

Thus, $a \leq b \Rightarrow c(a) \geq c(b)$

$\therefore$ c is a fuzzy compliment.

<c_3> clearly c is continuous.

<c_4> $c(c(a)) = c\left(\frac{1}{2}(1+\cos \pi a)\right)$

$= \frac{1}{2}\left(1+\cos \pi\left(\frac{1}{2}+\cos \pi a\right)\right)$

$c(c(a)) \neq a$

Hence c is not involuative.

Hence c is a continuous fuzzy compliment but not involuative.

* Class of fuzzy compliment :-

① A function $c:[0,1] \to [0,1]$ defined by

$\boxed{c_\lambda(a) = \frac{1-a}{1+\lambda a}}$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $\lambda \in [0,\infty]$

is called sugeno class of fuzzy compliment

② A function $c_\omega:[0,1] \to [0,1]$ defined by

$\boxed{c_\omega(a) = (1-a^\omega)^{1/\omega}}$, $\omega > 0$

is called Yager's class of fuzzy compliment.

* Example :-

1) For $\lambda \geq 0$. show that sugeno's class of fuzzy compliment is continuous & involuative.

$\Rightarrow$ we have a function $c:[0,1] \to [0,1]$ defined by,

$c_\lambda(a) = c(a) = \frac{1-a}{1+\lambda a}$, $\lambda \in [0,\infty]$

is called sugeno's class of fuzzy compliment

<c_1> $c(0) = \frac{1-0}{1+\lambda(0)} = \frac{1}{1} = 1$

& $c(1) = \frac{1-1}{1+\lambda(1)} = \frac{0}{1+\lambda} = 0$

$\therefore c(0)=1$ & $c(1)=0$

<c_2> let $a \leq b$

$\Rightarrow -a \geq -b$

$\Rightarrow 1-a \geq 1-b$

also, since, $a \leq b$

$\Rightarrow \lambda a \leq \lambda b$

$\Rightarrow 1+\lambda a \leq 1+\lambda b$

$\Rightarrow \frac{1}{1+\lambda a} \geq \frac{1}{1+\lambda b}$

Then, $\frac{1-a}{1+\lambda a} \geq \frac{1-b}{1+\lambda b}$

$\Rightarrow c(a) \geq c(b)$

Thus, $a \leq b \Rightarrow c(a) \geq c(b)$

$\therefore$ $c(a)$ is a fuzzy compliment.

<c_3> we have,

$c(a) = \frac{1-a}{1+\lambda a}$, $\lambda \geq 0$

Then, $1+\lambda a \neq 0$, $\forall a$

$\therefore$ c is continuous.

<c_4> for any $a \in [0,1]$

consider, $c(c(a)) = c\left(\frac{1-a}{1+\lambda a}\right)$

$= \dfrac{1-\left(\frac{1-a}{1+\lambda a}\right)}{1+\lambda\left(\frac{1-a}{1+\lambda a}\right)}$

$$= \frac{1+\lambda a - 1 + a}{1+\lambda a + \lambda - \lambda a}$$

$$= \frac{\lambda a + a}{1+\lambda}$$

$$= \frac{a(\lambda+1)}{(1+\lambda)}$$

$c(c(a)) = a$

Thus, sugeno class of fuzzy compliment is continuous & involuative.

2) Show that Yagers class of fuzzy compliment is continuous & involuative.

$\Rightarrow$ we have a function $c : [0,1] \longrightarrow [0,1]$ defined by,

$c_w(a) = (1-a^w)^{1/w}, \quad w>0$

is called Yager's class of fuzzy compliment.

$\langle c_1 \rangle$ $c_w(0) = (1-(0)^w)^{1/w} = (1-0)^{1/w} = (1)^{1/w} = 1$

& $c_w(1) = (1-(1)^w)^{1/w} = (1-1)^{1/w} = (0)^{1/w} = 0$

$\therefore c_w(0) = 1, \quad c_w(1) = 0$

$\langle c_2 \rangle$ let $a \leq b$

$\cancel{-a \geq -b}$

$\cancel{1-a \geq 1-b}$

$\cancel{a \leq b}$

$a^w \leq b^w$

$-a^w \geq -b^w$

$1-a^w \geq 1-b^w$

$(1-a^w)^{1/w} \geq (1-b^w)^{1/w}$

$c_w(a) \geq c_w(b)$

Thus, $a \leq b \Rightarrow c_w(a) \geq c_w(b)$

$\therefore c_w(a)$ is fuzzy compliment.

$\langle c_3 \rangle$ $c_w(a) = (1-a^w)^{1/w}, \quad w>0$

this is an exponential function

$\therefore c$ is continuous

$\langle c_4 \rangle$ for any $a \in [0,1]$

consider,

$c_w(c_w(a)) = c_w((1-a^w)^{1/w})$

$= [1-((1-a^w)^{1/w})^w]^{1/w}$

$= 1-((1-a)^w)^{1/w}$

$= [1-1+a^w]^{1/w}$

$= [a^w]^{1/w} = a$

$\therefore c_w(c_w(a)) = a$

Thus, Yager's class of fuzzy compliment is continuous & involuative.

* **Equilibrium point :-**

Def^n: let $c : [0,1] \longrightarrow [0,1]$ be a fuzzy compliment and element $a \in [0,1]$ is called equilibrium point if $\boxed{c(a) = a}$

§ Example:-

1) Find the equilibrium point for fuzzy compliment c, defined by $c(a) = 1-a$

$\Rightarrow$ by def^n of equilibrium point, $c(a) = a$

$c(a) = a$

$1-a = a$

$\Rightarrow 1 = a+a$

$\Rightarrow 1 = 2a$

$\Rightarrow \boxed{a = \frac{1}{2}}$

hence $a = 1/2$ is an equilibrium point of c

2) Find the equilibrium point for fuzzy compliment c, defined by $c(a) = \frac{1}{2}(1+\cos \pi a)$

$1+\cos\pi\theta = 2\cos^2\left(\frac{\theta\pi}{2}\right)$

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→ A fuzzy compliment c is defined by,

$$c(a) = \frac{1}{2}(1+\cos\pi a)$$

we have if e is an equilibrium point of c then,

$$c(e) = e$$

$$\frac{1}{2}(1+\cos\pi e) = e$$

$$1+\cos\pi e = 2e$$

$$2\cos^2\left(\frac{\pi e}{2}\right) = 2e$$

$$\cos^2\left(\frac{\pi e}{2}\right) = e$$

$$\cos\left(\frac{\pi e}{2}\right) = \sqrt{e}$$

since $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$\cos\left(\frac{\pi}{2}\cdot\frac{1}{2}\right) = \sqrt{\frac{1}{2}}$$

hence, $e = \frac{1}{2}$

Thus, $e = \frac{1}{2}$ is an equilibrium point for given fuzzy compliment c

3) find equilibrium point for sugeno class of fuzzy compliment.

→ let $C_\lambda : [0,1] \to [0,1]$ be a function then sugeno's class of fuzzy compliment C_λ is defined as,

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$$C_\lambda(a) = \frac{1-a}{1+\lambda a}, \quad \lambda \in [0,\infty]$$

if e is an equilibrium point of C_λ, then

$$C_\lambda(e) = e$$

$$\frac{1-e}{1+\lambda e} = e$$

$$1-e = e(1+\lambda e)$$

$$1-e = e + e\lambda e$$

$$1-e-e = \lambda e^2$$

$$1-2e = \lambda e^2$$

$$1 = \lambda e^2 + 2e$$

$$\lambda e^2 + 2e - 1 = 0$$

$$a = \lambda,\ b = 2,\ c = -1$$

$$e = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$e = \frac{-2 \pm \sqrt{2^2-4(\lambda)(-1)}}{2\lambda}$$

$$e = \frac{-2 \pm \sqrt{4+4\lambda}}{2\lambda}$$

$$e = \frac{-2 \pm \sqrt{4(1+\lambda)}}{2\lambda}$$

$$e = \frac{-2 \pm 2\sqrt{1+\lambda}}{2\lambda}$$

$$e = \frac{-1 \pm \sqrt{1+\lambda}}{\lambda}$$

$$e = \frac{-1+\sqrt{1+\lambda}}{\lambda} \text{ or } \cancel{e = \frac{-1-\sqrt{1+\lambda}}{\lambda}}$$

Since, $\lambda \geq 0$ & $e \in [0,1]$

Then $e = \frac{-1+\sqrt{1+\lambda}}{\lambda}$

Thus, $e = \frac{-1+\sqrt{1+\lambda}}{\lambda}$ is an equilibrium point for sugeno's class of fuzzy compliment.

Q) Find the equilibrium point of Yagers class of fuzzy compliment.

$\Rightarrow$ let $C_w : [0,1] \rightarrow [0,1]$ be a function then Yager's class of fuzzy compliment.

C_w is defined as,

$$C_w(a) = (1-a^w)^{1/w} \quad , \quad w>0$$

If e is equilibrium point of C_w then,

$$C_w(e) = e$$

$$(1-e^w)^{1/w} = e$$

$$(1-e^w) = e^w$$

$$1-e^w = e^w$$

$$1 = e^w + e^w$$

$$1 = 2e^w$$

$$e^w = \left(\frac{1}{2}\right)^{1/w}$$

$$e = \left(\frac{1}{2}\right)^{1/w}$$

or $e = 2^{-1/w}$

Thus, $e = \left(\frac{1}{2}\right)^{1/w}$ is an equilibrium point of Yager's class of fuzzy compliment.

Theorem :-

Every fuzzy compliment has atmost one equilibrium point.

proof :-

let 'c' be a fuzzy compliment

If 'a' is an equilibrium point of c then

$$c(a) = a \quad , \quad a \in [0,1]$$

i.e, $c(a) - a = 0 \quad a \in [0,1]$

In general, we have to show that, the eq^n of the type,

$c(a) - a = b$ has atmost one solution

on the contrary suppose that,

a_1 & a_2 are two different sol^n of the eq^n $c(a) - a = b$

then, $c(a_1) - a_1 = b$

& $c(a_2) - a_2 = b$

$\Rightarrow c(a_1) - a_1 = c(a_2) - a_2$

If $a_1 < a_2$

$\Rightarrow c(a_1) > c(a_2)$

Also, $a_1 < a_2$

$\Rightarrow -a_1 > -a_2$

$\Rightarrow c(a_1) \underset{-a_1}{>} c(a_2) - a_2$

$\Rightarrow c(a_1) - a_1 > c(a_2) - a_2$

$\Rightarrow b > b$

which is contradiction.

Hence eq^n $c(a) - a = b$ can not have two solutions.

i.e, If solution is exist then it must be unique

In particular for $b = 0$

we have,

$c(a) - a = 0$

$\Rightarrow$ clearly, this eq^n has atmost one solution

i.e, every fuzzy compliment has atmost one equilibrium point.

Hence the proof.

* Theorem :-

Assume that, the fuzzy compliment c has equilibrium point e_c then

① $a \leq c(a)$ iff $a \leq e_c$

② $a \geq c(a)$ iff $a \geq e_c$

proof :-

If e_c is an equilibrium point of c then

$c(e_c) = e_c$

by above th^m,

e_c must be unique.

Now, for any $a \in [0,1]$, we have

either $a < e_c$ or $a = e_c$ or $a > e_c$

Now, $a < e_c$

$\Rightarrow c(a) > c(e_c) = e_c$

$\Rightarrow c(a) > e_c$

$\Rightarrow c(a) > a$

i.e, $a < c(a)$

Next, $a = e_c$

$\Rightarrow c(a) = c(e_c)$

$\Rightarrow c(a) = e_c = a$

$\Rightarrow c(a) = a$

Now, $a > e_c$

$\Rightarrow c(a) < c(e_c) = e_c$

$\Rightarrow c(a) < e_c < a$

$\Rightarrow c(a) < a$

Thus, we have

$a \leq e_c \Rightarrow a \leq c(a)$

$a \geq e_c \Rightarrow a \geq c(a)$

Conversely,

For any $a \in [0,1]$

$\therefore$ we have,

either, $a < c(a)$ or $a = c(a)$ or $a > c(a)$

Now, if $a < c(a)$ or $a > c(a)$

then $e_c \neq a$

$\therefore$ either $e_c < a$ or $e_c > a$

If $a < c(a)$ & $e_c < a$ holds then

$c(e_c) > c(a)$

$\Rightarrow e_c > c(a) > a$

which is contradiction

hence, $a < c(a) \Rightarrow a < e_c$

Similarly, $c(a) < a$ & $e_c > a$ holds

then $c(e_c) < c(a)$

$\Rightarrow c(e_c) < c(a)$

$\Rightarrow e_c < c(a) < a$

$\Rightarrow e_c < a$

which is contradiction

Hence $a > c(a)$

$\Rightarrow e_c < a$

& $c(a) = a$

$\Rightarrow a = e_c$

hence, $a < c(a) \Rightarrow a \leq e_c$

$a \geq c(a) \Rightarrow a \geq e_c$

Thus, $a \leq c(a) \Leftrightarrow a \leq e_c$

& $a \geq c(a) \Leftrightarrow a \geq e_c$

Hence the proof.

* Dual Element (dual point)

Def^n :-

let c be fuzzy compliment & $a \in [0,1]$ the element $d_a \in [0,1]$ is called dual point of 'a' with r.t. 'c'

If $\boxed{c(d_a) - d_a = a - c(a)}$

OR :- A point is called dual element of 'a' if

$c(d_a) - d_a = a - c(a)$

Example :-

1) If c is a standard fuzzy compliment then for any $a \in [0,1]$ & if d_a is a dual element

⟹ Solution:

we know,

$c(d_a) - d_a = a - c(a)$

$\Rightarrow$ ~~$c(d_a) - d_a = a$~~

$\Rightarrow 1 - d_a - d_a = a - (1-a)$

$\Rightarrow 1 - 2d_a = 2a - 1$

$\Rightarrow -2d_a = 2a - 1 - 1$

$\Rightarrow -2d_a = 2a - 2$

$\Rightarrow 2d_a = 2 - 2a = 2(1-a)$

$\Rightarrow d_a = 1 - a$

$\Rightarrow d_a = c(a)$

Thus for standard fuzzy compliment

$d_a = c(a) = 1 - a \quad , \forall a \in [0,1]$

* Note :- If e_c is an equilibrium point then $d_{e_c} = c(e_c) = e_c$

@ If c is continuous fuzzy compliment then c has unique eq^m point

* Theorem :-

If c is any fuzzy compliment which has an equilibrium point e_c then $d_{e_c} = e_c$

proof :-

If e_c is an equilibrium point then

$c(e_c) = e_c$

If d_{e_c} is a dual of e_c then we have,

$c(d_{e_c}) - d_{e_c} = e_c - c(e_c)$

$c(d_{e_c}) - d_{e_c} = e_c - e_c$

$c(d_{e_c}) - d_{e_c} = 0$

$\Rightarrow c(d_{e_c}) = d_{e_c}$

$\Rightarrow d_{e_c}$ is an equilibrium point.

$\therefore$ ~~$d_{e_c} \sim$~~

but equilibrium point is unique.

hence, $d_{e_c} = e_c$.

Hence the proof.

* Theorem :-

For each $a \in [0,1]$, $d_a = c(a)$ iff $c(c(a)) = a$

OR :- prove that dual of an element is it's compliment iff c is ~~involutive~~ involuative.

proof :-

let $d_a = c(a) \quad \forall a \in [0,1]$

then $c(d_a) = c(c(a))$

by def^n of dual element.

we have,

$c(d_a) - d_a = a - c(a)$

$\Rightarrow c(c(a)) - \cancel{d_a}\, c(a) = a - c(a)$

$\Rightarrow c(c(a)) = a$

i.e, C is involutive.
conversely,
suppose that, C is involutive,
i.e, $c(c(a)) = a$
$\Rightarrow$ $c(c(a)) - c(a) = a - c(a)$
$\Rightarrow$ $c(a)$ is a dual of a
$\Rightarrow$ $d_a = c(a)$
Thus $d_a = c(a)$ iff $c(c(a)) = a$
Hence the proof.

* Note:
since sugeno's class & yagre's class of fuzzy compliment are involutive.
The above thm says that, the dual element of his compliment w.r.t. sugeno's class & Yagare's class.

** Increasing generator function :-
Def^n:
A continuous function $g: [0,1] \to R$ s.t. $\boxed{g(0)=0}$ & g is strictly increasing is called increasing generator function.

Example:
let $g: [0,1] \longrightarrow R$ defined by $g(a) = a$ then g is continuous function,
$g(0)=0$ & $a < b \Rightarrow g(a) < g(b)$.
hence g is an increasing generator function.

§ Involutive fuzzy compliment :
def^n: let $g: [0,1] \longrightarrow R$ be an increasing generator function, the function
$c: [0,1] \longrightarrow [0,1]$ defined by,
$c(a) = g^{-1}(g(1) - g(a))$.
is an involutive fuzzy compliment.

* Example :-
let $g: [0,1] \longrightarrow R$ defined by,
$g(a) = a$
be the increasing generator.
let $c: [0,1] \longrightarrow [0,1]$ be a function defined by $c(a) = g^{-1}(g(1) - g(a))$
since, $g(a) = a$
$\Rightarrow$ $g^{-1}(a) = a$
$g(1) = 1 \Rightarrow g^{-1}(1) = 1$
$\therefore$ $c(a) = g^{-1}(g(1) - g(a))$
$c(a) = g^{-1}(1 - a)$
$\Rightarrow$ $c(a) = 1 - a$
which is standard fuzzy compliment.

* Examples :-
1) let $g: [0,1] \longrightarrow R$ be a function defined by $g_\lambda(a) = \frac{1}{\lambda}\log(1+\lambda a)$, $\lambda > 0$.
show that, g_λ is an increasing generator & obtained ~~involutive~~ fuzzy compliment generated by g_λ.
$\Rightarrow$ let $g: [0,1] \to R$ be a function defined by
$g_\lambda(a) = \frac{1}{\lambda}\log(1+\lambda a)$, $\lambda > 0$.
Clearly, g_λ is continuous, now
$g_\lambda(0) = \frac{1}{\lambda}\log(1+\lambda(0))$

$$g_\lambda(0) = \frac{1}{\lambda}\log(1)$$

$$\boxed{g_\lambda(0) = 0} \qquad (\because \log 1 = 0)$$

let, $a < b$

$\lambda a < \lambda b$

$(1+\lambda a) < (1+\lambda b)$

$\log(1+\lambda a) < \log(1+\lambda b)$

$\frac{1}{\lambda}\log(1+\lambda a) < \frac{1}{\lambda}\log(1+\lambda b)$

$g_\lambda(a) < g_\lambda(b)$

Thus, $a<b \Rightarrow g_\lambda(a) < g_\lambda(b)$

i.e, g_λ is strictly increasing ~~generator~~

hence g_λ is an increasing generator.

since we have,

$g_\lambda(a) = \frac{1}{\lambda}\log(1+\lambda a)$, $\lambda > 0$

let $g_\lambda(a) = b$

than $a = g_\lambda^{-1}(b)$

Now $g_\lambda(a) = b$

$\frac{1}{\lambda}\log(1+\lambda a) = b$

$\log(1+\lambda a) = \lambda b$

$\Rightarrow (1+\lambda a) = e^{\lambda b}$

$\Rightarrow \lambda a = e^{\lambda b} - 1$

$\Rightarrow a = \frac{e^{\lambda b}-1}{\lambda}$

but, $a = g_\lambda^{-1}(b)$

but $g_\lambda^{-1}(b) = \frac{e^{\lambda b}-1}{\lambda}$

$g_\lambda^{-1}(b) = \frac{e^{\lambda b}-1}{\lambda}$ $\qquad a \in [0,1]$

Now, define a fun^n

$c : [0,1] \longrightarrow [0,1]$ by

$c(a) = g_\lambda^{-1}(g_\lambda(1) - g_\lambda(a))$

$c(a) = g_\lambda^{-1}\left(\frac{1}{\lambda}\log(1+\lambda) - \frac{1}{\lambda}\log(1+\lambda a)\right)$

$c(a) = g_\lambda^{-1}\left(\frac{1}{\lambda}(\log(1+\lambda) - \log(1+\lambda a)\right)$

$c(a) = g_\lambda^{-1}\left(\frac{1}{\lambda}\log\left(\frac{1+\lambda}{1+\lambda a}\right)\right]$

$c(a) = \frac{e^{\lambda\left(\frac{1}{\lambda}\log\left(\frac{1+\lambda}{1+\lambda a}\right)\right)}-1}{\lambda}$

$= \frac{e^{\log\left(\frac{1+\lambda}{1+\lambda a}\right)}-1}{\lambda}$

$= \frac{\frac{1+\lambda}{1+\lambda a}-1}{\lambda}$

$= \frac{1+\lambda \cancel{-\lambda a} - 1(1+\lambda a)}{\lambda(1+\lambda a)}$

$= 1+\lambda$

$= \frac{1+\lambda-1-\lambda a}{\lambda(1+\lambda a)} = \frac{\lambda(1-a)}{\lambda(1+\lambda a)}$

$= \frac{1-a}{1+\lambda a}$

which is a sugeno class of fuzzy compliment.

& $g_\lambda(a) = \frac{1}{\lambda}\log(1+\lambda a)$ is the generator of sugeno class of fuzzy compliment.

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2) Show that a function $g_\omega : [0,1] \to R$ defined by, $g_\omega(a) = a^\omega$, $\omega > 0$ is an increasing generator for Yager's class of fuzzy compliment.

$\Rightarrow$ let, $g_\omega : [0,1] \longrightarrow R$ defined by,

$g_\omega(a) = a^\omega$, $\omega > 0$

clearly, g_ω is continuous, now

$g_\omega(0) = a^0$

$\boxed{g_\omega(0) = 0}$

let, $a < b$

$a^\omega < b^\omega$

$g_\omega(a) < g_\omega(b)$

Thus, $a < b \rightarrow g_\omega(a) < g_\omega(b)$

i.e, g_ω is strictly increasing.

hence, g_ω is increasing generator.

since, we have

$g_\omega(a) = a^\omega$, $\omega > 0$

let $g_\omega(a) = b$

than $a = g_\omega^{-1}(b)$

Now $g_\omega(a) = b$

$a^\omega = b$

$a = b^{1/\omega}$.

but $a = g_\omega^{-1}(b)$

$g_\omega^{-1}(b) = b^{1/\omega}$

$g_\omega^{-1}(a) = a^{1/\omega}$ $a \in [0,1]$

Now define funⁿ,

$c : [0,1] \longrightarrow [0,1]$

$c(a) = g_\omega^{-1}(g_\omega(1) - g_\omega(a))$

$c(a) = g_\omega^{-1}((1)^\omega - a^\omega)$

$c(a) = g_\omega^{-1}(1 - a^\omega)$

$= (1-a^\omega)^{1/\omega}$

$= (1-a^\omega)^{1/\omega}$

which is a Yager's class of fuzzy compliment.

$\therefore$ $g_\omega(a) = a^\omega$, ~~$\omega > 0$~~ is generator of Yager's class of fuzzy compliment.

3) Show that the function

$g_\gamma(a) = \frac{a}{\gamma + (1-\gamma)a}$, $\gamma > 0$ is an increasing generator & find the class of fuzzy compliment generated by g_γ

$\Rightarrow$ let, $g_\gamma(a) = \frac{a}{\gamma + (1-\gamma)a}$, $\gamma > 0$

clearly, g_γ is continuous, now

$g_\gamma(0) = \frac{0}{\gamma + (1-\gamma)0}$

$\boxed{g_\gamma(0) = 0}$

let $a < b$

$g_\gamma(a) < g_\gamma(b)$

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$\Rightarrow (1-\gamma)a < (1-\gamma)b$

$\Rightarrow \gamma + (1-\gamma)a < \gamma + (1-\gamma)b$

$\Rightarrow \dfrac{a}{\gamma+(1-\gamma)a} < \dfrac{b}{\gamma+(1-\gamma)b}$

$\Rightarrow g_\gamma(a) < g_\gamma(b)$

Thus, $a<b \Rightarrow g_\gamma(a) < g_\gamma(b)$

i.e. g_γ is strictly increasing.

Hence, g_γ is increasing generator.

since we have

$$g_\gamma(a) = \frac{a}{\gamma+(1-\gamma)a}, \quad \gamma>0$$

let, $g_\gamma(a) = b$

then, $a = g_\gamma^{-1}(b)$

Now, $g_\gamma(a) = b$

$$\frac{a}{\gamma+(1-\gamma)a} = b$$

$$a = b(\gamma+(1-\gamma)a)$$

$$a = b(\gamma + a - \gamma a)$$

$$a = b\gamma + ba - b\gamma a$$

$$a + b\gamma a = b\gamma + ba$$

$$a(1+b\gamma) = b\gamma + ba$$

$$a = \frac{b(\gamma+a)}{(1+b\gamma)}$$

but $a = g_\gamma^{-1}(b)$

$$g_\gamma^{-1}(b) = \frac{b(\gamma+a)}{1+b\gamma}$$

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$$a + b\gamma a - ba = b\gamma$$

$$a(1+b\gamma-b) = b\gamma$$

$a(1-(1-\gamma)b)$

$$a = \frac{b\gamma}{1-(1-\gamma)b}$$

but, $a = g_\gamma^{-1}(b)$

$$g_\gamma^{-1}(b) = \frac{b\gamma}{1-(1-\gamma)b}$$

$$g_\gamma^{-1}(a) = \frac{a\gamma}{1-(1-\gamma)a} \qquad a \in [0,1]$$

Now define funⁿ

$c : [0,1] \to [0,1]$

$$c(a) = g_\gamma^{-1}(g_\gamma(1) - g_\gamma(a))$$

$$c(a) = g_\gamma^{-1}\left(\frac{1}{\gamma+(1-\gamma)} - \frac{a}{\gamma+(1-\gamma)a}\right)$$

$$c(a) = g^{-1}_\gamma\left(1 - \frac{a}{\gamma+(1-\gamma)a}\right)$$

$$= g^{-1}_\gamma\left(\frac{\gamma+(1-\gamma)a - a}{\gamma+(1-\gamma)a}\right)$$

$$= g^{-1}_\gamma\left(\frac{\gamma+\cancel{a}-\gamma a-\cancel{a}}{\gamma+(1-\gamma)a}\right)$$

$$= g^{-1}_\gamma\left(\frac{\gamma(1-a)}{\gamma+(1-\gamma)a}\right)$$

$\left(\cancel{\gamma+(1-\gamma)a}\right.$

$$= g^{-1}_\gamma\left(\frac{\gamma+(1-\gamma)a - a(\cancel{\gamma+(1-\gamma)})}{(\cancel{\gamma+(1-\gamma)})(\gamma+(1-\gamma)a)}\right)$$

$$= g^{-1}_\gamma\left(\frac{\gamma+\cancel{a}-\gamma a - a\gamma - \cancel{a} + a\gamma}{(\cancel{\gamma+(1-\gamma)})^2\, a}\right)$$

$$= \cancel{g^{-1}_\gamma\left(\frac{\gamma - a\gamma}{(\gamma+(1-\gamma))^2 a}\right)}$$

$$= g^{-1}_\gamma\left(\frac{\gamma(1-a)}{\gamma+(1-\gamma)a}\right)$$

$$= \frac{\left(\frac{\gamma(1-a)}{\cdots}\right)\gamma}{1-\cancel{(1-\gamma}\left(\frac{\gamma(1-a)}{a}\right)}$$

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$$\frac{\dfrac{\gamma(1-a)}{\gamma+(1-\gamma)a}}{1-(1-\gamma)\left[\dfrac{\gamma(1-a)}{\gamma+(1-\gamma)a}\right]}$$

$$= \frac{\dfrac{\gamma^2(1-a)}{\gamma+(1-\gamma)a}}{\dfrac{\gamma+(1-\gamma)a-(1-\gamma)[\gamma(1-a)]}{\gamma+(1-\gamma)a}}$$

$$= \frac{\gamma^2(1-a)}{\gamma+(1-\gamma)a-(1-\gamma)\gamma(1-a)}$$

$$= \frac{\gamma^2(1-a)}{\gamma+a-\gamma a-\gamma(1-\gamma)(\gamma-a\gamma)}$$

$$= \frac{\gamma^2(1-a)}{\gamma+a-\gamma a-\gamma+a\gamma+\gamma^2-a\gamma^2}$$

$$= \frac{\gamma^2(1-a)}{a+\gamma^2-a\gamma^2}$$

$$\boxed{c(a) = \frac{\gamma^2(1-a)}{a+\gamma^2(1-a)}}$$

which is required fuzzy compliment.

* decreasing generator function

A function $f: [0,1] \longrightarrow \mathbb{R}$ which is continuous strictly decreasing & $\boxed{f(1)=0}$ is called a decreasing generator.

* Theorem:

let $c: [0,1] \longrightarrow [0,1]$ be a function then c is a fuzzy compliment iff $\exists$ a continuous function $f: [0,1] \longrightarrow \mathbb{R}$, which is decreasing generator & $c(a)$ defined by $c(a)$:

$c(a) = f^{-1}(f(0)-f(a))$

proof :-

let $c: [0,1] \longrightarrow [0,1]$

be a fuzzy compliment

then $\exists$ an increasing generator g s.t.

$c(a) = g^{-1}(g(1)-g(a))$ & $g(0)=0$

Now, define a function $f: [0,1] \longrightarrow \mathbb{R}$ by

$f(a) = g(1) - g(a)$.

Now, we have to prove that f is decreasing generator.

since, g is continuous,

f is also continuous.

Next, $f(1) = g(1) - g(1)$

$f(1) = 0$

if $a < b$

$\Rightarrow g(a) < g(b)$

$\Rightarrow -g(a) > -g(b)$

$\Rightarrow g(1) - g(a) > g(1) - g(b)$

$\Rightarrow f(a) > f(b)$

Thus, $a<b \Rightarrow f(a) > f(b)$

hence f is strictly decreasing.

Now if $f(a) = b$

$\Rightarrow a = f^{-1}(b)$

but $f(a) = b$

$g(1) - g(a) = b$

$\Rightarrow \quad g(a) = g(1) - b$

$\Rightarrow \quad a = g^{-1}(g(1) - b)$

$\Rightarrow \quad f^{-1}(b) = g^{-1}(g(1) - b)$

$\Rightarrow \quad f^{-1}(a) = g^{-1}(g(1) - a) \quad \text{--- (1)}$

Also, $f(a) = g(1) - g(a)$

$f(0) = g(1) - g(0)$

$f(0) = g(1)$

$\therefore \quad f^{-1}(a) = g^{-1}(f(0) - a) \quad (g(1) = f(0)) \quad *$

Now, $c(a) = g^{-1}(g(1) - g(a))$

$= g^{-1}(f(0) - g(a))$

$= f^{-1}(g(a)) \quad$ (by ①)

$= \cancel{f^{-1}(g(1) - b)}$

$= f^{-1}(g(1) - f(a))$

$= f^{-1}(g(1) - f(a))$

$c(a) = f^{-1}(f(0) - f(a))$

Conversely, suppose that, $\exists$ a decreasing generator function f, s.t. $f(1) = 0$ & $c(a) = f^{-1}(f(0) - f(a))$

we have to prove that, c is fuzzy compliment.

<c1> we have,

$c(a) = f^{-1}(f(0) - f(a))$

$c(0) = f^{-1}(f(0) - f(0))$

$= f^{-1}(0) \quad (f(1) = 0 \Rightarrow 1 = f^{-1}(0))$

$c(0) = 1$

& $c(1) = f^{-1}(f(0) - f(1))$

$= f^{-1}(f(0) - 0)$

$= f^{-1}(f(0))$

$c(1) = 0$

<c2>

let $a_1 < a_2$

$f(a_1) > f(a_2)$

$-f(a_1) < -f(a_2)$

$f(0) - f(a_1) < f(0) - f(a_2)$

$f^{-1}(f(0) - f(a_1)) > f^{-1}(f(0) - f(a_2))$

$c(a_1) > c(a_2)$

Thus, $a_1 < a_2 \Rightarrow c(a_1) > c(a_2)$

$\therefore$ c is fuzzy compliment.

<c3> Further,

$c(a) = f^{-1}(f(0) - f(a))$

since f & f^{-1} are continuous

$\therefore$ c is also continuous.

<c4> $c(c(a)) = a$

consider, $c(c(a)) = c(f^{-1}(f(0) - f(a)))$

$= f^{-1}(f(0) - f(f^{-1}(f(0) - f(a))))$

$= f^{-1}(f(0) - (f(0) - f(a)))$

$= f^{-1}(f(0) - f(0) - f(a))$

$= f^{-1}(f(a))$

$= a$

$c(c(a)) = a$

Thus c is an involutive compliment.

Hence the proof.

* Fuzzy Intersection :- (t - norm)

A fuzzy intersection or a t-norm i is a binary operation $i : [0,1] \times [0,1] \to [0,1]$ with satisfies following axioms.

1) Boundary cond^n

$i(a,1) = a \quad \forall \ a \in [0,1]$

② Monotone
If $b \le d$ then $i(a,b) \le i(a,d)$, $\forall a \in [0,1]$

③ Commutative
$i(a,b) = i(b,a)$

④ Associative
$i(i(a,b),d) = i(a, i(b,d))$

If i satisfies all above four properties then i is called a t-norm operator.
Further t-norm operator i may satisfies following axioms

⑤ Continuity
i is continuous function

⑥ Sub-idempotent
$i(a,a) < a$ $\forall a \in [0,1]$

⑦ i is strictly monotone
if $a_1 < a_2$ & $b_1 < b_2$ then
$i(a_1,b_1) < i(a_2,b_2)$

* Note :-
A t-norm i which is continuous & subidempotent is called an archimedean t-norm.
Further if it is strictly monotone then it is called strict archimedean t-norm

Example :-
The following are examples of t-norm.
① standard intersection
$i(a,b) = \min(a,b)$

② Algebraic product
$i(a,b) = a\,b$

③ Bounded difference
$i(a,b) = \max(0, a+b-1)$

④ Drastic intersection
$$i(a,b) = \begin{cases} a & \text{if } b=1 \\ b & \text{if } a=1 \\ 0 & \text{otherwise} \end{cases}$$

This t-norm is denoted by i_{min}

* Theorem :-
The standard fuzzy intersection is the only t-norm which is idempotent.
proof :-
we have the standard fuzzy intersection is defined by,
$A \cap B(x) = \min(A(x), B(x))$
let $i = \min$
i.e. $i(a,b) = \min(a,b)$
1) let $i(a,1) = \min(a,1)$ $\forall a \in [0,1]$
$i(a,1) = a$
i.e. boundary condition is satisfied

i_2) let, $b \le d$

$\Rightarrow \min(a,b) \le \min(a,d)$

$\Rightarrow i(a,b) \le i(a,d)$

$\Rightarrow i$ is monotone

i_3) commutative

since, $\min(a,b) = \min(b,a)$

$\Rightarrow i(a,b) = i(b,a)$

$\Rightarrow i$ is commutative

i_4) Associative

$i(a, i(b,d))$

$= \min(a, \min(b,d))$

$= a \wedge (b \wedge d)$

$= (a \wedge b) \wedge d$

$= \min(\min(a,b), d)$

$= i(i(a,b), d)$

ie, i is associative

Thus, $i(a,b) = \min(a,b)$ is a t-norm

also, $i(a,a) = \min(a,a)$

$= a$

ie, i is idempotent

Now, let i' be any other t-norm s.t.

$i'(a,a) = a$

then form $a,b \in [0,1]$ with $a \le b \le 1$

$a = i'(a,a)$

$\le i'(a,b)$

$\le i'(a,1)$

$a \le a$

ie, $i'(a,b) = a$

also, $\min(a,b) = a$

$\Rightarrow i'(a,b) = \min(a,b) = a$

Similarly,

If $b \le a \le 1$ then

$\min(a,b) = b$

& $b = i'(b,b)$

$\le i'(b,a)$

$\le i'(b,1)$

$b \le b$

$\Rightarrow i'(b,a) = b$

$\therefore i'(a,b) = b$ by commutative

$\therefore i'(a,b) = \min(a,b) = b$

hence $i' = \min$

Thus min is the only idempotent t-norm

Hence the proof.

* Theorem :-

For all $a,b \in [0,1]$ then

$i_{min}(a,b) \le i(a,b) \le \min(a,b)$

where, i is the t-norm, i_{min} is the drastic intersection t-norm

proof :-

let i be any other t-norm with which satisfied. i_1, i_2, i_3, i_4

then for all $b \in [0,1]$

$\Rightarrow b \le 1$

$i(a,b) \le i(a,1) = a$ $\forall\, a \in [0,1]$

$\Rightarrow i(a,b) \le a$

Similarly,

$\forall\, a \in [0,1]$

$\Rightarrow a \le 1$

$i(a,b) \leq i(1,b) = b \quad \forall\ b \in [0,1]$

$\Rightarrow i(a,b) \leq b$

$\therefore i(a,b) \leq a$ & $i(a,b) \leq b$: $\forall\ a,b \in [0,1]$

$\Rightarrow i(a,b) \leq \min(a,b)$ ——— (*)

Next. for any $a \in [0,1]$

$\Rightarrow 0 \leq a$

$\Rightarrow i(b,0) \leq i(b,a) = i(a,b)$

but $i(b,0) \leq \min(b,0) = 0$ $\quad$ ($\because$ by (*))

$\Rightarrow i(b,0) = 0$

Thus, $0 \leq i(a,b) \quad \forall\ a,b \in [0,1]$

Thus,

$$i(a,b) = i(a,1) = a \quad , \text{ if } b = 1$$
$$= i(1,b) = b \quad , \text{ if } a = 1$$
$$\geq 0 \quad \text{otherwise}$$

But $i_{min} = a$ if $b=1$

$= b$ if $a=1$

$= 0$ otherwise. $\quad$ (**)

hence, $i_{min}(a,b) \leq i(a,b) \quad \forall\ a,b \in [0,1]$

$i_{min}(a,b) \leq i(a,b) \leq \min(a,b)$

Hence the proof.

* Pseudo inverse of decreasing generator:

Defn:-

Let $f: [0,1] \to \mathbb{R}$ be a decreasing generator. The pseudo inverse of f denoted by $f^{(-1)}$ is a function denoted by $f^{(-1)}: \mathbb{R} \to [0,1]$ defined by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [0, f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

OR :-

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [f(1), f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

* Example :-

1) Let $f: [0,1] \to \mathbb{R}$ be a function defined by $f(a) = 1 - a^p$, $p > 0$. Show that f is decreasing generator & find it's pseudo inverse.

$\Rightarrow$ let $f: [0,1] \to \mathbb{R}$ be a function defined by,

$$f(a) = 1 - a^p, \quad p > 0$$

clearly, f is continuous &

$f(1) = 1 - 1^p = 1 - 1 = 0$

also, for $a < b$, $a,b \in [0,1]$

$$\Rightarrow a^p < b^p$$
$$\Rightarrow -a^p > -b^p$$
$$\Rightarrow 1 - a^p > 1 - b^p$$
$$\Rightarrow f(a) > f(b)$$

Thus, $a < b \Rightarrow f(a) > f(b)$

f is strictly decreasing

hence f is decreasing generator.

Next,

The pseudo inverse of f is $f^{(-1)}: \mathbb{R} \to [0,1]$ defined by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [0, f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

let $f(a) = b$
then $a = f^{-1}(b)$
Now, $f(a) = b$

$$1 - a^p = b$$
$$a^p = 1 - b$$
$$a = (1-b)^{1/p}$$
$$f^{-1}(b) = (1-b)^{1/p}$$
$$\Rightarrow f^{-1}(a) = (1-a)^{1/p}$$

hence pseudo inverse of f is given by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ (1-a)^{1/p} & \text{if } a \in [0, 1] \\ 0 & \text{if } a \in (1, \infty) \end{cases}$$

Q. Show that the function $f : [0,1] \to \mathbb{R}$ defined by $f(a) = -\ln(a)$, $a \in [0,1]$ & $f(0) = \infty$ is a decreasing generator & find its pseudo inverse.

$\Rightarrow$ let $f : [0,1] \longrightarrow \mathbb{R}$
defined by,

$$f(a) = -\ln(a), \quad a \in [0,1]$$
& $f(0) = \infty$

Clearly,
f is continuous
& $f(1) = -\ln(1)$
$= -\log_e(1)$
$f(1) = 0$

Now, for any $x, y \in [0,1]$
Let $x < y$

$\Rightarrow \log_e x < \log_e y$
$\Rightarrow -\log_e x > -\log_e y$
$\Rightarrow -\ln(x) > -\ln(y)$
$\Rightarrow f(x) > f(y)$

Thus, $x < y \Rightarrow f(x) > f(y)$
Thus the function f is strictly decreasing
hence f is decreasing generator
Next,
The pseudo inverse of f is,
$f^{(-1)}(a) : \mathbb{R} \to [0,1]$ defined by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [f(1), f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

but $f(0) = \infty$
then pseudo inverse of f is,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [0, \infty) \end{cases}$$

Now, $f(a) = b$
$a = f^{-1}(b)$
$f(a) = b$
$\Rightarrow -\ln(a) = b$
$\Rightarrow -\log_e(a) = b$
$\Rightarrow \log_e(a) = -b$
$\Rightarrow a = e^{-b}$
$\Rightarrow f^{-1}(b) = e^{-b}$
$\Rightarrow f^{-1}(a) = e^{-a}$

hence pseudo inverse is,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ e^{-a} & \text{if } a \in [0, \infty) \end{cases}$$

Pseudo inverse of increasing generator

Def^n:-
A continuous & strictly increasing function $g: [0,1] \to \mathbb{R}$ s.t. $g(0)=0$ is called an increasing generator.
The pseudo inverse of an increasing generator g is given by
$g^{(-1)}: \mathbb{R} \to [0,1]$ defined by,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in [g(1), \infty) \end{cases}$$

Ex 1) Show that a function $g: [0,1] \to \mathbb{R}$ defined by $g(a) = a^p$, $p>0$ is an increasing generator & find its pseudo inverse

Solution:-
let a function $g: [0,1] \longrightarrow \mathbb{R}$ defined by
$g(a) = a^p$, $p>0$
clearly, g is continuous
& $g(0) = 0^p = 0$
& $g(1) = 1^p = 1$
Next,
for any $x, y \in [0,1]$
let $x < y$
$\Rightarrow x^p < y^p$
$\Rightarrow g(x) < g(y)$
Thus $x<y \Rightarrow g(x) < g(y)$
$\therefore g$ is strictly increasing
hence the function g is an increasing generator.
let, $g^{(-1)}: \mathbb{R} \longrightarrow [0,1]$
~~&~~ be pseudo inverse of g defined by,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in [g(1), \infty) \end{cases}$$

~~Now,~~
let $g(a) = b$
$\Rightarrow a = g^{-1}(b)$
but $g(a) = b$
$a^p = b$
$a = b^{1/p}$
$g^{-1}(b) = b^{1/p}$
$g^{-1}(a) = a^{1/p}$
hence pseudo inverse is,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ a^{1/p} & \text{if } a \in [0, 1] \\ 1 & \text{if } a \in (1, \infty) \end{cases}$$

Ex 2) Show that the function $g: [0,1] \to \mathbb{R}$ defined by $g(a) = -\ln(1-a)$, $a \in [0,1]$ & $g(1) = \infty$ is an increasing generator & find its pseudo inverse.

Sol^n:-
let the fun^n $g: [0,1] \longrightarrow \mathbb{R}$ defined by
$g(a) = -\ln(1-a)$, $a \in [0,1]$ & $g(1) = \infty$
clearly, g is continuous
$g(0) = -\ln(1-0) = 0 - \log_e(1) = 0$
i.e. $g(0) = 0$
Now,
for any $x, y \in [0,1]$
let $x < y$
$-x > -y$
$1-x > 1-y$
$\log_e(1-x) > \log_e(1-y)$

$-\ln(1-x) < -\ln(1-y)$

$g(x) < g(y)$

Thus, $x<y \Rightarrow g(x) \le g(y)$

$\therefore$ g is strictly increasing.

hence the function g is an increasing generator

let, $g^{(-1)} : \mathbb{R} \to [0,1]$

be pseudo inverse defined by,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

let $g(a) = b$

$a = g^{-1}(b)$

but $g(a) = b$

$-\ln(1-a) = b$

$-\log_e(1-a) = b$

$\log_e(1-a) = -b$

$(1-a) = e^{-b}$

$a = 1 - e^{-b}$

$g^{-1}(b) = 1 - e^{-b}$

$g^{-1}(a) = 1 - e^{-a}$

hence pseudo inverse is,

$$g^{(-1)}(a) = \begin{cases} 0 & , \text{ if } a \in (-\infty, 0) \\ 1-e^{-a} & , \text{ if } a \in [0, \infty) \end{cases}$$

Theorem :-

let f be a decreasing generator then the function g defined by,

$g(a) = f(0) - f(a)$, $a \in [0,1]$ is an increasing generator with $g(1) = f(0)$ & its pseudo inverse is given by,

$g^{(-1)}(a) = f^{(-1)}(f(0) - a)$, $\forall a \in \mathbb{R}$.

proof :-

let f be a decreasing generator

$\therefore$ f is strictly decreasing, continuous

& $f(1) = 0$

Now, we define a function g by,

$g(a) = f(0) - f(a)$, $a \in [0,1]$

since, we defined a g in terms of f

hence, g is also continuous

& $g(0) = f(0) - f(0) = 0$

$\Rightarrow g(0) = 0$

$g(1) = f(0) - f(1) = f(0) - 0 = f(0)$

$g(1) = f(0)$

For any $x, y \in [0,1]$,

let $x<y$

$\Rightarrow f(x) > f(y)$

$\Rightarrow -f(x) < -f(y)$

$\Rightarrow f(0) - f(x) < f(0) - f(y)$

$\Rightarrow g(x) < g(y)$

Thus, $x<y \Rightarrow g(x) < g(y)$

$\therefore$ g is strictly increasing.

hence, g is an increasing generator.

pseudo inverse of g is given by,

$$g^{(-1)}(a) = \begin{cases} 0 & , \text{ if } a \in (-\infty, 0) \\ g^{-1}(a) & , \text{ if } a \in [0, g(1)] \\ 1 & , \text{ if } a \in (g(1), \infty) \end{cases}$$

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If $a \in [0,1]$ then,

$g(a) = b$

$\Rightarrow a = g^{-1}(b)$

but $g(a) = f(0) - f(a)$

$f(0) - f(a) = b$

$\Rightarrow f(a) = f(0) - b$

$\Rightarrow a = f^{-1}(f(0)-b)$

$\Rightarrow g^{-1}(b) = f^{-1}(f(0)-b)$

$\Rightarrow g^{-1}(a) = f^{-1}(f(0)-a)$

pseudo inverse defined by,

$$g^{(-1)}(a) = \begin{cases} 0, & \text{if } a \in (-\infty, 0) \\ f^{-1}(f(0)-a), & \text{if } a \in [0, \overset{g(1)}{\cancel{f(0)}}] \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

i.e.,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{, if } a \in (-\infty, 0) \\ f^{-1}(f(0)-a) & \text{if } a \in [0, f(0)] \\ 1 & \text{if } a \in (f(0), \infty) \end{cases} \quad \text{——} (*)$$

Also, f is a decreasing generator pseudo inverse of f is given by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [0, f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

Replace 'a' by $f(0)-a$

$$f^{(-1)}(f(0)-a) = \begin{cases} 1 & \text{, if } (f(0)-a) \in (-\infty, \overset{0}{\cancel{f(0)}}) \\ f^{-1}(f(0)-a) & \text{, if } (f(0)-a) \in [0, f(0)] \\ 0 & \text{, if } (f(0)-a) \in (f(0), \infty) \end{cases}$$

if, $f(0)-a \in (-\infty, \overset{0}{\cancel{f(0)-a}})$

$\Rightarrow -\infty < f(0)-a < 0$

$\Rightarrow -\infty - f(0) < -a < 0 - f(0)$

$\Rightarrow -\infty < -a < -f(0)$

$\Rightarrow \infty > a > f(0)$

$\Rightarrow f(0) < a < \infty$

$\Rightarrow a \in (f(0), \infty)$

if, $f(0)-a \in [0, f(0)]$

$\Rightarrow 0 \le f(0)-a \le f(0)$

$\Rightarrow 0-f(0) \le -a \le f(0)-f(0)$

$\Rightarrow -f(0) \le -a \le 0$

$\Rightarrow f(0) \ge a \ge 0$

$\Rightarrow 0 \le a \le f(0)$

$\Rightarrow a \in [0, f(0)]$

if $f(0)-a \in (f(0), \infty)$

$\Rightarrow f(0) < f(0)-a < \infty$

$\Rightarrow f(0)-f(0) < -a < \infty - f(0)$

$\Rightarrow 0 < -a < \infty$

$\Rightarrow 0 > a > -\infty$

$\Rightarrow -\infty < a < 0$

$\Rightarrow a \in (-\infty, 0)$

Thus we have,

$$f^{(-1)}(f(0)-a) = \begin{cases} 1 & \text{, if } a \in (f(0), \infty) \\ f^{-1}(f(0)-a), & \text{if } a \in [0, f(0)] \\ 0 & \text{, if } a \in (-\infty, 0) \end{cases}$$

$$f^{(-1)}(f(0)-a) = \begin{cases} 0 & \text{, if } a \in (-\infty, 0) \\ f^{-1}(f(0)-a) & \text{, if } a \in [0, f(0)] \\ 1 & \text{, if } a \in [f(0), \infty) \end{cases} \quad \text{——} (**)$$

$\therefore$ from eq^n (*) & (**)

we have,

$g^{(-1)}(a) = f^{(-1)}(f(0), a) \quad \forall a \in R.$

hence the proof.

Theorem

let g be an increasing generator then the fun^n f defined by $f(a) = g(1) - g(a)$, $a \in [0,1]$ is an decreasing function with $f(0) = g(1)$ & it pseudo inverse is given by $f^{(-1)}(a) = g^{(-1)}(g(1) - a)$

imp

proof:

given g is an increasing generator.

$\therefore$ g is strictly increasing, continuous & $g(0) = 0$

Now, define a function f by,

$f(a) = g(1) - g(a)$, $a \in [0,1]$

since, f is defined in terms g hence f is continuous

Now, $f(1) = g(1) - g(1) = 0$

$\Rightarrow f(1) = 0$

For any $x, y \in [0,1]$

let $x < y$

$\Rightarrow g(x) < g(y)$

$\Rightarrow -g(x) > -g(y)$

$\Rightarrow g(1) - g(x) > g(1) - g(y)$

$\Rightarrow f(x) > f(y)$

Thus $x < y \Rightarrow f(x) > f(y)$

$\therefore$ f is strictly decreasing

hence, f is decreasing generator function

Now, pseudo inverse of f is given by,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f^{-1}(a) & \text{if } a \in [0, f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

If $a \in [0,1]$ then,

$f(a) = b$

$\Rightarrow a = f^{-1}(b)$

but, $f(a) = b$

$g(1) - g(a) = b$

$\Rightarrow g(a) = g(1) - b$

$\Rightarrow a = g^{-1}(g(1) - b)$

$\Rightarrow f^{-1}(b) = g^{-1}(g(1) - b)$

$\Rightarrow f^{-1}(a) = g^{-1}(g(1) - a)$

we have,

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ g^{-1}(g(1) - a) & \text{if } a \in [0, f(0)] \\ 0 & \text{if } a \in (f(0), \infty) \end{cases}$$

since,

$f(0) = g(1)$

$$f^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ g^{-1}(g(1) - a) & \text{if } a \in [0, g(1)] \\ 0 & \text{if } a \in (g(1), \infty) \end{cases} \quad \text{—(}$$

Also, g is increasing generator

The pseudo inverse of g is given by,

$$g^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g^{-1}(a) & \text{if } a \in [0, g(1)) \\ 1 & \text{if } a \in (g(1), \infty) \end{cases}$$

Replace 'a' by $g(1) - a$

$$g^{(-1)}(g(1)-a) = \begin{cases} 0 & \text{if } (g(1)-a) \in (-\infty, 0) \\ g^{-1}(g(1)-a) & \text{if } (g(1)-a) \in [0, g(1)] \\ 1 & \text{if } (g(1)-a) \in (g(1), \infty) \end{cases}$$

if $g(1)-a \in (-\infty, 0)$

$\Rightarrow -\infty < g(1)-a < 0$

$\Rightarrow -\infty - g(1) < -a < 0 - g(1)$

$\Rightarrow -\infty < -a < -g(1)$

$\Rightarrow \infty > a > g(1)$

$\Rightarrow g(1) < a < \infty$

$\Rightarrow a \in (g(1), \infty)$

if $g(1)-a \in [0, g(1)]$

$\Rightarrow 0 \leq g(1)-a \leq g(1)$

$\Rightarrow 0 - g(1) \leq -a \leq g(1) - g(1)$

$\Rightarrow -g(1) \leq -a \leq 0$

$\Rightarrow g(1) \geq a \geq 0$

$\Rightarrow 0 \leq a \leq g(1)$

$\Rightarrow a \in [0, g(1)]$

if $g(1)-a \in (g(1), \infty)$

$\Rightarrow g(1) < g(1)-a < \infty$

$\Rightarrow g(1) - g(1) < -a < \infty - g(1)$

$\Rightarrow 0 < -a < \infty$

$\Rightarrow 0 > a > -\infty$

$\Rightarrow -\infty < a < 0$

$\Rightarrow a \in (-\infty, 0)$

Thus we have,

$$g^{(-1)}(g(1)-a) = \begin{cases} 0 & \text{if } a \in (g(1), \infty) \\ g^{-1}(g(1)-a) & \text{if } a \in [0, g(1)] \\ 1 & \text{if } a \in (-\infty, 0) \end{cases}$$

$$g^{(-1)}(g(1), a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ g^{-1}(g(1)-a) & \text{if } a \in [0, g(1)] \\ 0 & \text{if } a \in (g(1), \infty) \end{cases} \quad —(2)$$

From eq^n (1) & (2)

we have,

$f^{(-1)}(a) = g^{(-1)}(g(1)-a)$.

Hence the proof.

* Construction of t-norm

Def^n:-

let $i: [0,1] \times [0,1] \rightarrow [0,1]$ be a function then i is an archimedian of t-norm iff there exist a decreasing generator f, such that $i(a,b) = f^{(-1)}(f(a)+f(b)) \quad \forall\ a,b \in [0,1]$

6 mark

Ex. let $f_p: [0,1] \rightarrow \mathbb{R}$ be defined by $f_p(a) = 1 - a^p$, $a \in [0,1]$, $p>0$ show that f_p is decreasing generator & obtain t-norm generated by f_p.

Solution:-

let $f_p: [0,1] \rightarrow \mathbb{R}$ be defined by,

$f_p(a) = 1 - a^p$, $a \in [0,1]$, $p>0$

clearly, f_p is continuous also

also $f_p(1) = 1 - 1^p = 1-1 = 0$

$\Rightarrow f_p(1) = 0$

and $f_p(0) = 1 - (0)^p = 1-0 = 1$

$\Rightarrow f_p(0) = 1$

Next,

for any $a, b \in [0,1]$

let $a < b$

$a^p < b^p$

$$-a^p > -b^p$$
$$1-a^p > 1-b^p$$
$$f_p(a) > f_p(b)$$

Thus, $a<b \Rightarrow f_p(a) > f_p(b)$

f_p is strictly decreasing

hence f_p is decreasing generator function

The pseudo inverse of f_p is given by,

$$f_p^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ f_p^{-1}(a) & \text{if } a \in [0, f_p(0)] \\ 0 & \text{if } a \in (f_p(0), \infty) \end{cases}$$

let $f_p(a) = b$

$\Rightarrow a = f_p^{-1}(b)$

but we have,

$\Rightarrow f_p(a) = b$

$\Rightarrow 1-a^p = b$

$\Rightarrow a^p = 1-b$

$\Rightarrow a = (1-b)^{1/p}$

$\Rightarrow f_p^{-1}(b) = (1-b)^{1/p}$

$\Rightarrow f_p^{-1}(a) = (1-a)^{1/p}$

hence pseudo inverse of f_p is,

$$f_p^{(-1)}(a) = \begin{cases} 1 & \text{if } a \in (-\infty, 0) \\ (1-a)^{1/p} & \text{if } a \in [0,1] \\ 0 & \text{if } a \in (1, \infty) \end{cases}$$

define, $i : [0,1] \times [0,1] \longrightarrow [0,1]$ by

$$i(a,b) = f_p^{(-1)}(f_p(a) + f_p(b))$$
$$= f_p^{(-1)}(1-a^p+1-b^p)$$
$$i(a,b) = f_p^{(-1)}(2-a^p-b^p)$$

then $$i(a,b) = \begin{cases} 1 & \text{, if } 2-a^p-b^p \in (-\infty, 0) \\ (1-(2-a^p-b^p))^{1/p} & \text{, if } 2-a^p-b^p \in [0,1] \\ 0 & \text{, if } 2-a^p-b^p \in (1,\infty) \end{cases}$$

$$i(a,b) = \begin{cases} 1 & \text{, if } 2-a^p-b^p \in (-\infty, 0) \\ (a^p+b^p-1)^{1/p} & \text{, if } 2-a^p-b^p \in [0,1] \\ 0 & \text{, if } 2-a^p-b^p \in (1,\infty) \end{cases}$$

$$i(a,b) = \begin{cases} (a^p+b^p-1)^{1/p} & \text{, if } 2-a^p-b^p \in [0,1] \\ 0 & \text{, if } 2-a^p-b^p \in (1,\infty) \end{cases}$$

If $2-a^p-b^p \in [0,1]$

$\Rightarrow 0 \le 2-a^p-b^p \le 1$

$\Rightarrow 0 \le 1-(2-a^p-b^p)$

$\Rightarrow 0 \le a^p+b^p-1$

& If $2-a^p-b^p \in (1,\infty)$

$\Rightarrow 2-a^p-b^p > 1$

$\Rightarrow 0 > 1-(2-a^p-b^p)$

Thus, $$i(a,b) = \begin{cases} (a^p+b^p-1)^{1/p} & \text{, if } 0 \le a^p+b^p-1 \\ 0 & \text{, if } a^p+b^p-1 < 0 \end{cases}$$

$$i(a,b) = \max(0, (a^p+b^p-1)^{1/p})$$

Note :- for $p=1$

$i(a,b) = \max(0, a+b-1)$

This is called a bounded difference t-norm

* t-norm generated by a function :-

Defn :- let i be a given t-norm & let $g : [0,1] \to [0,1]$ be a function which is strictly increasing & continuous in $[0,1]$ with $g(1) = 1$ & $g(0) = 0$ then the function i^g is defined by,

$$i^g(a,b) = g^{(-1)}(i(g(a), g(b))) \quad \forall\ a,b \in [0,1]$$

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then i^g is t-norm generated by g.

Ex) Obtain a t-norm generated by a function $g: [0,1] \to [0,1]$ by

$$g(a) = \begin{cases} \frac{a+1}{2} & , a \neq 0 \\ 0 & , a = 0 \end{cases}$$

& $i(a,b) = a.b$, $\forall\ a,b \in [0,1]$

Solution:-

$\Rightarrow$ given a function $g: [0,1] \to [0,1]$ defined by;

$$g(a) = \begin{cases} \frac{a+1}{2} & , a \neq 0 \\ 0 & ; a = 0 \end{cases}$$

clearly,

$g(0) = 0$ &

$g(1) = \frac{1+1}{2} = \frac{2}{2} = 1$

$g(1) = 1$

also, let, $a > b$

$\Rightarrow a+1 > b+1$

$\Rightarrow \frac{a+1}{2} > \frac{b+1}{2}$

$\Rightarrow g(a) > g(b)$

Thus, $a > b \Rightarrow g(a) > g(b)$

i.e. g is strictly increasing

clearly g is continuous

hence, g is increasing generator function.

let. $g(a) = b$

$\Rightarrow a = g^{-1}(b)$

we have,

$g(a) = b$

$\Rightarrow \frac{a+1}{2} = b$

$\Rightarrow a+1 = 2b$

$\Rightarrow a = 2b-1$

$g^{-1}(b) = 2b-1$

$g^{-1}(a) = 2a-1$

$\therefore$

$$g^{(-1)}(a) = \begin{cases} 0 & , \text{otherwise} \\ 2a-1 & , a \in (\frac{1}{2}, 1] \end{cases}$$

also given that,

$i(a,b) = a.b$

Now define a function

$$i^g(a,b) = g^{(-1)}(i(g(a), g(b)))$$

$$= g^{(-1)}(g(a) \cdot g(b))$$

$$= g^{(-1)}\left(\left(\frac{a+1}{2}\right)\left(\frac{b+1}{2}\right)\right)$$

$$i^g(a,b) = \begin{cases} 2\left(\frac{a+1}{2}\right)\left(\frac{b+1}{2}\right) - 1 & , \left(\frac{a+1}{2}\right)\left(\frac{b+1}{2}\right) \in \left(\frac{1}{2}, 1\right] \\ 0 & \text{otherwise} \end{cases}$$

$$i^g(a,b) = \begin{cases} \frac{ab+a+b+1}{2} - 1 & , \left(\frac{a+1}{2}\right)\left(\frac{b+1}{2}\right) \in \left(\frac{1}{2}, 1\right] \\ 0 & , \text{otherwise} \end{cases}$$

if $\left(\frac{a+1}{2}\right)\left(\frac{b+1}{2}\right) \in \left(\frac{1}{2}, 1\right]$

$\Rightarrow \frac{1}{2} < \left(\frac{a+1}{2}\right)\left(\frac{b+1}{2}\right) \leq 1$

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$$\Rightarrow \frac{1}{2} < \frac{ab+a+b+1}{4} \leq 1$$

$$\Rightarrow 1 \leq \frac{ab+a+b+1}{2} \leq 2$$

$$\Rightarrow 1-1 \leq \frac{ab+a+b+1}{2} - 1 \leq 2-1$$

$$\Rightarrow 0 \leq \frac{ab+a+b+1}{2} - 1 \leq 1$$

$$\Rightarrow 0 \leq \frac{ab+a+b+1-2}{2} \leq 1$$

$$\Rightarrow 0 \leq \frac{ab+a+b-1}{2} \leq 1$$

$\therefore$

$$i^{g}(a,b) = \begin{cases} \frac{ab+a+b+1}{2} - 1, & 0 \leq \frac{ab+a+b-1}{2} \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

$$i^{g}(a,b) = \max\left(0, \frac{ab+a+b+1}{2} - 1\right)$$

$$\boxed{i^{g}(a,b) = \max\left(0, \frac{ab+a+b-1}{2}\right)}$$

* Fuzzy t-conorm : (fuzzy union)

A fuzzy t-conorm u is binary operation on $[0,1]$ with satisfies following axioms

u_1> $u(a,0) = a$ (boundary condition)

u_2> $b \leq d \Rightarrow u(a,b) \leq u(a,d)$ (monotone)

u_3> $u(a,b) = u(b,a)$ (commutative)

u_4> $u(a, u(b,d)) = u(u(a,b), d)$ (Associative)

In addition the fun^n 'u' may satisfy the following axioms.

u_5> u is ~~a~~ continious.

u_6> $u(a,a) > a$ (super idempotent)

u_7> $a_1 < a_2$, $b_1 < b_2 \Rightarrow u(a_1, b_1) < u(a_2, b_2)$ - (strict monotone)

* Note :-

A t-conorm which is continious & super idempotent is called an Archimedian t-conorm. further it is strict monotone then it is called strict Archimedian t-conorm.

* Theorem:- 6 mark

The standard fuzzy union is the only idempotent t-conorm.

$\Rightarrow$ proof:-

The standard fuzzy union operator is given by

$(A \cup B)(x) = \max(A(x), B(x))$

i.e. $= A(x) \vee B(x)$

let $u = \max$

i.e. $u(a,b) = \max(a,b)$

then we have

u_1> $u(a,0) = \max(a,0) = a$

i.e. boundary cond^n is satisfy.

u_2) let $b \le d$
$\max(a,b) \le \max(a,d)$
$u(a,b) \le u(a,d)$
i.e. u is Monotone.

u_3) since $\max(a,b) = \max(b,a)$
$\Rightarrow$ $u(a,b) = u(b,a)$ i.e. u is commutative

u_4) $u(a, u(b,d))$
$= \max(a, u(b,d))$
$= \max(a, \max(b,d))$
$= a \vee \max(b,d)$
$= a \vee b \vee d$
$= (a \vee b) \vee d$
$= \max(a,b) \vee d$
$= \max(\max(a,b), d)$
$= u(u(a,b), d)$
i.e. u is associative
Thus, $u(a,b) = \max(a,b)$ is satisfy
u_1, u_2, u_3, u_4.
Hence $u(a,b) = \max(a,b)$ is a t-conorm.
Further for all $a \in [0,1]$
$u(a,a) = \max(a,a)$
$= a$
Hence, $u(a,b) = \max(a,b)$ is an ~~ideam~~ idem potent t-conorm
Next, if $\bar{u}$ is an idempotent t-conorm other than u then we have to show that,
$\bar{u} = \max$
let $a,b \in [0,1]$
& let $0 \le a \le b \le 1$
$b = \bar{u}(a,b)$
$= \bar{u}(b,a)$
$b \le \bar{u}(b,a)$
& $\bar{u}(b,a) \le \bar{u}(b,b)$
$\le b$
Thus, $b \le \bar{u}(b,a)$ & $\bar{u}(b,a) \le b$
$\Rightarrow b = \bar{u}(b,a)$
but we have, $a \le b$
i.e. $\max(a,b) = b$
Thus, $\bar{u}(a,b) = \max(a,b) = b$
Similarly, if $0 \le b < a \le 1$, we have
$\bar{u}(a,b) = \max(a,b) = a$
$\therefore$ for all $a,b \in [0,1]$
$\bar{u}(a,b) = \max(a,b)$
i.e. $\bar{u} = \max$
Thus the standard fuzzy union is the only idempotent t-conorm.
Hence the proof.

* Example
→ Following are the some example of t-conorm
1) Standard union.
$u(a,b) = \max(a,b)$
2) algebraic sum
$u(a,b) = a + b - ab$
3) bounded sum
$u(a,b) = \min(1, a+b)$
4) Drastic union
$$u(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{otherwise} \end{cases}$$
this t-conorm is denoted by u_{max}.

* Theorem :-

For all $a, b \in [0,1]$ then prove that,

$\max(a,b) \leq u(a,b) \leq U_{max}(a,b)$

~~since~~

proof :-

since u is t-conorm for all $a, b \in [0,1]$

with $0 \leq a$, $0 \leq b$

Then $a = u(a,0)$

$a \leq u(a,b)$

& $b = u(0,b)$

$b \leq u(a,b)$

Thus, we have $a \leq u(a,b)$ & $b \leq u(a,b)$

$\Rightarrow \max(a,b) \leq u(a,b)$ —— ① $\forall a,b \in [0,1]$

Next,

when $b = 0$, $u(a,b) = u(a,0) = a$

& when $a = 0$, $u(a,b) = u(0,b) = b$

also, $u(a,b) \leq 1$

Thus, $u(a,b) = a$ if $b = 0$

$= b$ if $a = 0$

≤ 1 otherwise

also,

$U_{max}(a,b) = a$ if $b = 0$

$= b$ if $a = 0$

$= 1$, otherwise

Hence,

$u(a,b) \leq U_{max}(a,b)$ —— ②, $\forall a,b \in [0,1]$

From eq^n ① & ②, we get

$\max(a,b) \leq u(a,b) \leq U_{max}(a,b)$, $\forall a,b \in [0,1]$

* Def^n:-

let u be a binary operation on $[0,1] \times [0,1] \to [0,1]$

then u is a archimedian t-conorm

iff $\exists$ increasing generator g s.t.

$u(a,b) = g^{(-1)}(g(a) + g(b))$

* Example

1) Show that $g_p : [0,1] \to \mathbb{R}$ defined by

$g_p(a) = 1 - (1-a)^p$, $p \neq 0$ is an increasing generator. find it's pseudo inverse & obtain a class of t-conorm generated by g_p

$\Rightarrow$ Given a function

$g_p : [0,1] \to \mathbb{R}$

defined by,

$g_p(a) = 1 - (1-a)^p$, $p \neq 0$

Here, $g_p(0) = 1 - (1-0)^p$

$= 1 - 1^p = 1 - 1$

$g_p(0) = 0$

since, g_p is an algebric function

& hence it is continuous.

let $a < b$, $a, b \in [0,1]$

$-a > -b$

$1-a > 1-b$

$(1-a)^p > (1-b)^p$

$-(1-a)^p < -(1-b)^p$

$1-(1-a)^p < 1-(1-b)^p$

$g_p(a) < g_p(b)$

Thus, $a < b \Rightarrow g_p(a) < g_p(b)$

$\therefore g_p$ is strictly increasing

hence g_p is increasing generator

The pseudo inverse of g_p is given by,

$$g_p^{(-1)}(a) = \begin{cases} 0 & \text{if } a \in (-\infty, 0) \\ g_p^{-1}(a) & \text{if } a \in [0, g_p(1)] \\ 1 & \text{if } a \in [g_p(1), \infty) \end{cases}$$

let $g_p(a) = b$

$\Rightarrow \quad a = g_p^{-1}(b)$

but $\quad g_p(a) = b$

$1-(1-a)^p = b$

$-(1-a)^p = b-1$

$(1-a)^p = 1-b$

$1-a = (1-b)^{1/p}$

$-a = (1-b)^{1/p} - 1$

$a = 1-(1-b)^{1/p}$

$\Rightarrow \quad g_p^{-1}(b) = 1-(1-b)^{1/p}$

$\Rightarrow \quad g_p^{-1}(a) = 1-(1-a)^{1/p}$

Next,

$g_p(1) = 1-(1-1)^p$

$= 1-(0)^p$

$g_p(1) = 1$

$$\therefore \quad g_p^{(-1)}(a) = \begin{cases} 0 & \text{if } (-\infty, 0) \\ 1-(1-a)^{1/p} & \text{if } [0,1] \\ 1 & \text{if } (1, \infty) \end{cases}$$

Define, t-conorm

$u : [0,1] \times [0,1] \longrightarrow [0,1]$

by $\quad u(a,b) = g_p^{(-1)}(g_p(a) + g_p(b))$

$= g_p^{(-1)}\left(1-(1-a)^p + 1-(1-b)^p\right)$

$= g_p^{(-1)}\left(2-(1-a)^p - (1-b)^p\right)$

$\forall \; a,b \in [0,1]$

but, $2-(1-a)^p - (1-b)^p \geq 0, \; \forall \; a,b \in [0,1]$

Hence, $2-(1-a)^p - (1-b)^p \notin (-\infty, 0)$

Hence $$u(a,b) = \begin{cases} 1-\left[1-\left(2-(1-a)^p-(1-b)^p\right)\right]^{1/p}, & \text{if } (2-(1-a)^p-(1-b)^p) \in [0,1] \\ 1 & \text{if } 2-(1-a)^p-(1-b)^p \in (1, \infty) \end{cases}$$

$$u(a,b) = \begin{cases} 1-\left[-1-(1-a)^p-(1-b)^p\right]^{1/p}, & \text{if } 2-(1-a)^p-(1-b)^p \in [0,1] \\ 1 & \text{if } 2-(1-a)^p-(1-b)^p \in (1, \infty) \end{cases}$$

$0 \leq 2-(1-a)^p-(1-b)^p \leq 1$

$\Rightarrow 0 \leq 1-(2-(1-a)^p-(1-b)^p)$.

$0 \leq (1-a)^p + (1-b)^p - 1$.

$$u(a,b) = \begin{cases} 1-\left[(1-a)^p+(1-b)^p-1\right]^{1/p}, & 0 \leq (1-a)^p+(1-b)^p-1 \\ 1 & \text{, otherwise} \end{cases}$$

i.e., $u(a,b) = 1-\max\{0, [(1-a)^p+(1-b)^p-1]^{1/p}\}$

This t-norm is called schweizer & scalar

[Note - for $p=1$

$u(a,b) = 1-\max(0, (1-a)+(1-b)-1)$

$= 1-\max(0, 1-a-b)$

$= 1-[1-a-b]$

$u(a,b) = 1-1+a+b = a+b$]

* Yager's class of t-conorm

Example:- If $g_\omega(a) = a^\omega$, $\omega > 0$ show that g_ω is an increasing generator & obtain a t-conorm generated by g_ω
$u_\omega(a,b) = \min\{1, (a^\omega + b^\omega)^{1/\omega}\}$, $\omega > 0$

→ Given

$g_\omega(a) = a^\omega$, $\omega > 0$

clearly, it is continuous

$g_\omega(0) = a$

let, for $a, b \in [0,1]$

let $a < b$

$a^\omega < b^\omega$

$g_\omega(a) < g_\omega(b)$

Thus, $a < b \Rightarrow g_\omega(a) < g_\omega(b)$

∴ g_ω is strictly increasing

hence, g_ω is increasing generator

The pseudo inverse of g_ω is given by

$$g_\omega^{(-1)}(a) = \begin{cases} 0 & , a \in (-\infty, 0) \\ g_\omega^{-1}(a) & , a \in [0, g_\omega(1)] \\ 1 & , a \in [g_\omega(1), \infty) \end{cases}$$

let $g_\omega(a) = b$

$a = g_\omega^{-1}(b)$

But $g_\omega(a) = b$

$a^\omega = b$

$a = b^{1/\omega}$

$g_\omega^{-1}(b) = b^{1/\omega}$

$\Rightarrow g_\omega^{-1}(a) = a^{1/\omega}$

& $g_\omega(1) = 1$

$$g_\omega^{(-1)}(a) = \begin{cases} 0 & , a \in (-\infty, 0) \\ a^{1/\omega} & , a \in [0,1] \\ 1 & , a \in (1, \infty) \end{cases}$$

Define,

t-conorm u by

$u_\omega(a,b) = g_\omega^{(-1)}(g_\omega(a) + g_\omega(b))$

$= g_\omega^{(-1)}(a^\omega + b^\omega)$

but $a^\omega + b^\omega \geq 0$

$a^\omega + b^\omega \notin (-\infty, 0)$

$$u_\omega(a,b) = \begin{cases} (a^\omega + b^\omega)^{1/\omega} & , \text{ if } (a^\omega + b^\omega) \in [0,1] \\ 1 & , \text{ if } (a^\omega + b^\omega) \in (1, \infty) \end{cases}$$

i.e., $u_\omega(a,b) = \min(1, (a^\omega + b^\omega)^{1/\omega})$, $\omega > 0$

This is called Yager's class of t-conorm.

* Note:-

If $\omega = 1$ then

$u_\omega(a,b) = \min\{1, (a+b)\}$

This is bounded sum t-conorm.

* Dual Triple: ⇒ IMP

A t-norm i & t-conorm u are said to be dual w.r.t. fuzzy compliment c if $c(u(a,b)) = i(c(a), c(b))$.

$c(i(a,b)) = u(c(a), c(b))$.

Then triple i, u, c is called a dual triple.

* Note :- $\langle i, u, c \rangle$ is called a dual triple if they satisfy demorgan's law.

1) Show that $\langle ab, a+b-ab, c\rangle$ is dual triple where, c is a std fuzzy compliment.

→ we have,

$i(a,b) = ab$

$u(a,b) = a+b-ab$

$c(a) =$ std fuzzy compliment $= 1-a$

consider,

① $c(u(a,b)) = i(c(a), c(b))$

$c(u(a,b)) = c(a+b-ab)$

$= 1-a-b+ab$

$= (1-a) - b(1-a)$

$= (1-a)(1-b)$

$= c(a)\,c(b)$

$= i(c(a), c(b))$ [∵ $i(a,b) = a.b$]

ie, $c(u(a,b)) = i(c(a), c(b))$ —— ①

② $c(i(a,b)) = u(c(a), c(b))$

consider, $c(i(a,b)) = c(a.b)$

$= 1-ab$ —— (*)

& $u(c(a), c(b)) = u(1-a, 1-b)$

$= 1-a+1-b-(1-a)(1-b)$

$= 2-a-b-(1-a-b+ab)$

$= 2-a-b-1+a+b-ab$

$= 1-ab$ —— (**)

from (*) & (**)

$c(i(a,b)) = u(c(a), c(b))$ —— ②

from ① & ②

$\langle ab, a+b-ab, c\rangle$ is a dual triple.

2) Show that $\langle \min, \max, c\rangle$ is a dual triple where c is std fuzzy compliment

→ let, $a, b \in [0,1]$

we have,

$i(a,b) = \min(a,b)$

$u(a,b) = \max(a,b)$

$c(a) =$ std fuzzy compliment $= 1-a$

Case: I>

when $a \le b$

consider,

$c(u(a,b)) = c(\max(a,b))$

$= c(b)$

$= 1-b$ —— ①

& $i(c(a), c(b)) = i(1-a, 1-b)$

$= \min(1-a, 1-b)$

$= 1-b$ —— ②

from ① & ②, we get

$c(u(a,b)) = i(c(a), c(b))$ —— (*)

Next,

consider,

$c(i(a,b)) = c(\min(a,b))$

$= c(a)$

$= 1-a$ —— ③

& $i(c(a), c(b)) = u(1-a, 1-b)$

$= \max(1-a, 1-b)$

$= 1-a$ —— ④

from ③ & ④

$c(i(a,b)) = u(c(a), c(b))$ —— (**)

Case - II

when $a \geq b$

consider,

$c(u(a,b)) = c(\max(a,b))$

$= c(a)$

$= 1-a$ —— ⑤

& $i(c(a), c(b)) = i(1-a, 1-b)$

$= \min(1-a, 1-b)$

$= 1-a$ —— ⑥

From ⑤ & ⑥

$i(u(a,b)) = (c(a), c(b))$ —— I

Next,

$c(i(a,b)) = c(\min(a,b))$

$= c(b)$

$= 1-b$ —— ⑦

$u(c(a), c(b)) = u(1-a, 1-b)$

$= \max(1-a, 1-b)$

$= 1-b$ —— ⑧

From ⑦ & ⑧

$c(i(a,b)) = u(c(a), c(b))$ —— II

hence from eqn (*) (**) & I & II,

we have,

$c(u(a,b)) = i(c(a), c(b))$ &

$c(i(a,b)) = u(c(a), c(b))$

hence, $<\min, \max, c>$ is a dual triple.

Theorem :-

let, i be a t-norm & c be involutive fuzzy compliment. The binary operation u defined on $[0,1]$ by,

$u(a,b) = c<i(c(a); c(b))>$ is a t-conorm s.t $<i, u, c>$ is a dual triple.

Given that,

$u : [0,1] \times [0,1] \longrightarrow [0,1]$

defined by,

$u(a,b) = c<i(c(a), c(b))>$

First we show that u is a t-conorm i.e, u satisfies u_1, u_2, u_3, u_4 axioms

u_1> Boundary condition.

i.e, $u(a,0) = c<i(c(a), c(0))>$

$= c<i(c(a), 1)>$

$= c<\min(c(a), 1)>$

$= c<c(a)>$

$= c(c(a))$

$= a$.

Thus, $u(a,0) = a \quad \forall a \in [0,1]$

u_2> Monotone :

let $a, b, d \in [0,1]$ s.t $b \leq d$

$\Rightarrow c(b) \geq c(d)$

$\Rightarrow i(c(a), c(b)) \geq i(c(a), c(d))$

$\Rightarrow c<i(c(a), c(b))> \leq c<i(c(a), c(d))>$

$u(a,b) \leq u(a,d)$

Thus u is monotone.

u_3> Commutative :-

let $u(a,b) = c<i(c(a), c(b))>$

$= c<i(c(b), c(a))>$

$= u(b,a)$

Thus u is commutative.

u_4) Associative

consider, for $a, b, d \in [0,1]$

we have,

$u(a, u(b,d)) = c\langle i(c(a), c(u(b,d)))\rangle$

$= c\langle i(c(a), c[c\langle i(c(b), c(d))\rangle])\rangle$

$= c\langle i(c(a), i(c(b), c(d)))\rangle$

$= c\langle i(i(c(a), c(b)), c(d))\rangle$

$= c\langle i(c(c\langle i(c(a), c(b))\rangle), c(d))\rangle$

i.e., $= c\langle i(c(u(a,b)), c(d))\rangle$

$= u(u(a,b), d)$

Thus,

$u(a, u(b,d)) = u(u(a,b), d)$

i.e, u is associative

$\therefore u$ satisfies u_1, u_2, u_3, u_4

hence, u is a t-conorm

Next,

we show that $\langle i, u, c\rangle$ is dual triple.

Consider,

$c(u(a,b)) = c(c\langle i(c(a), c(b))\rangle)$

$c(u(a,b)) = i(c(a), c(b))$ —— ①

&

$u(c(a), c(b)) = c\langle i(c(c(a)), c(c(b)))\rangle$

$u(c(a), c(b)) = c(i(a,b))$ —— ②

from eqⁿ ① & ②

$\langle i, u, c\rangle$ is a dual triple.

Hence the proof.

Example.

1) show that $\left\langle ab, \frac{a+b}{ab+1}, \frac{1-a}{1+a}\right\rangle$ is a dual triple.

Solⁿ

we have,

$i(a,b) = ab$

$u(a,b) = \frac{a+b}{ab+1}$

& $c(a) = \frac{1-a}{1+a}$

1) consider, $c(u(a,b)) = i(c(a), c(b))$

$c(u(a,b)) = c\left(\frac{a+b}{ab+1}\right)$

$= \dfrac{1 - \left(\frac{a+b}{ab+1}\right)}{1 + \left(\frac{a+b}{ab+1}\right)}$

$= \frac{(ab+1) - (a+b)}{ab+1+a+b}$

$c(u(a,b)) = \frac{ab+1-a-b}{ab+1+a+b}$ —— ①

$i(c(a), c(b)) = \left(\frac{1-a}{1+a}\right)\left(\frac{1-b}{1+b}\right)$

$= \frac{(1-a)(1-b)}{(1+a)(1+b)}$

$i(c(a), c(b)) = \frac{1+ab-a-b}{1+b+a+ab}$ —— ②

from ① & ②

$c(u(a,b)) = i(c(a), c(b))$

2) Consider,

$$u(c(a), c(b)) = c(i(a,b))$$

$$u(c(a), c(b)) = u\left(\frac{1-a}{1+a}, \frac{1-b}{1+b}\right)$$

$$c(i(a,b)) = c(ab)$$

$$c(i(a,b)) = \frac{1-ab}{1+ab} \quad \text{——— (1)}$$

$$u(c(a), c(b)) = u\left(\frac{1-a}{1+a}, \frac{1-b}{1+b}\right)$$

$$= \frac{\left(\frac{1-a}{1+a}\right) + \left(\frac{1-b}{1+b}\right)}{\left(\frac{1-a}{1+a}\right)\left(\frac{1-b}{1+b}\right) + 1}$$

$$= \frac{\frac{(1-a)(1+b) + (1+a)(1-b)}{1+b+a+ab}}{\frac{1+ab-a+b}{1+b+a+ab} + 1}$$

$$= \frac{\frac{1+\cancel{b}-\cancel{a}-ab+1-\cancel{b}+\cancel{a}-ab}{1+b+a+ab}}{\frac{1+ab-a+\cancel{b}+1+b+a+ab}{1+b+a+ab}}$$

$$= \frac{2-2ab}{2+2ab}$$

$$= \frac{1-ab}{1+ab} \quad \text{——— (2)}$$

from (1) & (2)

$$u(c(a), c(b)) = c(i(a,b))$$

From (1) & (2),

$\left\langle ab, \frac{a+b}{ab+1}, \frac{1-a}{1+a} \right\rangle$ is a dual triple.

3) let $i(a,b) = ab$ & c_λ be the sugeno's class of fuzzy compliment & obtained t-conorm u s.t. $\langle i, u, c \rangle$ is dual triple.

⇒ given,

$i(a,b) = ab$ be a t-norm & involutive fuzzy compliment c is a sugeno's class of fuzzy compliment.

i.e. $c_\lambda(a) = \frac{1-a}{1+\lambda a}$, $\lambda > 0$

Now, define t-conorm u by,

$$u(a,b) = c(i(c(a), c(b)))$$

$$\therefore u(a,b) = c_\lambda(i(c_\lambda(a), c_\lambda(b)))$$

$$= c_\lambda(c_\lambda(a) \cdot c_\lambda(b))$$

$$= c_\lambda\left(\left(\frac{1-a}{1+\lambda a}\right) \cdot \left(\frac{1-b}{1+\lambda b}\right)\right)$$

$$= \frac{1 - \left(\frac{1-a}{1+\lambda a}\right)\left(\frac{1-b}{1+\lambda b}\right)}{1 + \lambda\left[\left(\frac{1-a}{1+\lambda a}\right)\left(\frac{1-b}{1+\lambda b}\right)\right]}$$

$$= \frac{1 - \left(\frac{1-b-a+ab}{1+\lambda b+\lambda a+\lambda^2 ab}\right)}{1 + \lambda\left[\frac{1-b+a+ab}{1+\lambda b+\lambda a+\lambda^2 ab}\right]}$$

$$= \frac{\frac{1+\lambda b+\lambda a+\lambda^2 ab-1+b+a-ab}{1+\lambda b+\lambda a+\lambda^2 ab}}{\frac{1+\lambda b+\lambda a+\lambda^2 ab+\lambda-b\lambda+\lambda a+\lambda ab}{1+\lambda b+\lambda a+\lambda^2 ab}}$$

$$= \frac{\lambda^2 ab+2\lambda a+\cancel{\lambda b}+b+a-ab}{\lambda^2 ab+\lambda ab+\cancel{\lambda a}+\cancel{\lambda b}\ 2\lambda a+\lambda+1}$$

$$= \frac{\cancel{\lambda a b+\ldots+2a)}+b+a-ab}{\cancel{\lambda(\ldots)}}$$

$$= \frac{\lambda^2 ab + \lambda a + \lambda b + a + b - ab}{\lambda^2 ab + \lambda ab + \lambda + 1}$$

$$= \frac{\lambda(\lambda ab + a + b) + a + b - ab}{(1+\lambda) + (\lambda^2+\lambda)ab}$$

$$= \frac{(1+\lambda)a + (1+\lambda)b - (1-\lambda^2)ab}{(1+\lambda) + (\lambda^2+\lambda)ab}$$

$$= \frac{(1+\lambda)(a+b) - (1-\lambda^2)ab}{(1+\lambda) + \lambda(1+\lambda)ab}$$

$$= \frac{(1+\lambda)(a+b) - (1-\lambda)(1+\lambda)ab}{(1+\lambda)(1+\lambda ab)}$$

$$= \frac{(1+\lambda)[(a+b) - (1-\lambda)ab]}{(1+\lambda)(1+\lambda ab)}$$

$$= \frac{[(a+b) - (1-\lambda)ab]}{(1+\lambda ab)}$$

then by th^m,

$\langle i, u, c\rangle$ forms a dual triple.

i.e. $\left\langle ab, \frac{a+b-(1-\lambda)ab}{1+\lambda ab}, \frac{1-a}{1+\lambda a}\right\rangle$ forms a dual triple

* Note

1) In above for $\lambda = 0$,
$\langle ab, a+b-ab, 1-a\rangle$ is a dual triple.

2) In above for $\lambda = 1$
$\left\langle ab, \frac{a+b}{1+ab}, \frac{1-a}{1+a}\right\rangle$ is a dual triple.

Q) Show that $\langle \max(0, a+b-1), \min(1, a+b), c\rangle$ is dual triple, c is std. fuzzy complement.

→ Let, $i(a,b) = \max(0, a+b-1)$
$u(a,b) = \min(1, a+b)$
$c = 1-a$

$c(i(a,b)) = c(\max(0, a+b-1))$
$= 1 - \max(0, a+b-1)$
$= \min(1, 2-a-b)$
$= 2-a-b$

$u(c(a), c(b)) = u(1-a, 1-b)$
$= \min(1, 2-a-b)$
$= 2-a-b$

$\therefore c(i(a,b)) = u(c(a), c(b))$ —— (1)

Similarly,

$c(u(a,b)) = c(\min(1, a+b))$
$= 1 - \min(1, a+b)$
$= \max(0, 1-a-b)$
$= 0$

$i(c(a), c(b)) = i(1-a, 1-b)$
$= \max(0, 2-a-b-1)$
$= \max(0, 1-a-b)$
$= 0$

$c(u(a,b)) = i(c(a), c(b))$ —— (2)

From eq^n (1) & (2).

$\langle \max(0, a+b-1), \min(1, a+b), c\rangle$ is dual triple