Linear Equation with variable coefficient

A linear differential eq^n of order n with variable coefficient is eq^n of the form

$$a_0(x)y^n + a_1(x)y^{n-1} + \dots + a_n(x)y = b(x)$$

where, $a_0, a_1, a_2, \dots, a_n$ & b are complex value function on some interval I.

* **Singular Point :-**

The point where $a_0(x) = 0$ are called singular point.

Here, after we assume that $a_0(x) \neq 0$ on I therefore differential eq^n with variable coefficient is given by

$$y^n + \frac{a_1(x)}{a_0(x)} y^{n-1} + \dots + \frac{a_n(x)}{a_0(x)} y = \frac{b(x)}{a_0(x)}, \quad a_0(x) \neq 0$$

$\Rightarrow y^n + a_1(x)y^{n-1} + \dots + a_n(x)y = b(x)$

$\therefore$ The differential eq^n with variable coefficient can be written as $L(y) = b(x)$.

If $b(x) = 0$, for all $x \in I$ then we called $L(y) = 0$ as homogeneous eq^n with variable coefficient.

If $b(x) \neq 0$, $\forall x \in I$ then we called $L(y) = b(x)$ as non-homogeneous eq^n with variable coefficient.

Here, we assume that the function $a_1, a_2, \dots, a_n$ are continuous complex valued function on some real interval I.

* **Existance Theorem :-**

Let $a_1, a_2, \dots, a_n$ be continuous function on an interval I containg the points x_0 if $\alpha_1, \alpha_2, \dots \alpha_n$ are n constant. Then there exist solution ϕ of $L(y) = 0$.

$$L(y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)y = 0$$

satisfing $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, \dots, \phi^{n-1}(x_0) = \alpha_n$

* **Result :-**

1) Continuous function define on an interval I need not be bounded.

2) Continuous function define on closed & bounded interval atends its bound.

3) Close bounded interval is of the form $a \leq x \leq b$ with a & b are real.

4) If I is close bounded interval & if the function $a_j(x)$ are the continuous on I then there always exist finite constant b_j such that $|a_j(x)| \leq b_j$ on I.

* **Theorem :-**

Let $b_1, b_2, \dots, b_n$ be non-negative constant such that for all $x \in I$ such that

$|a_j(x)| \leq b_j$ for $j = 1, 2, \dots, n$

$K = 1 + b_1 + b_2 + \dots + b_n$

If x_0 is a point in I & ϕ is a solution of $L(y) = 0$ then $\|\phi(x_0)\| e^{-K|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{K|x-x_0|}$

where, $\|\phi(x)\| = [|\phi(x)|^2 + |\phi'(x)|^2 + \dots + |\phi^{n-1}(x)|^2]^{1/2}$ —*

Proof :-

Let $u(x) = \|\phi(x)\|^2$

i.e. $u = \|\phi(x)\|^2$

$u = |\phi(x)|^2 + |\phi'(x)|^2 + \dots + |\phi^{n-1}(x)|^2$

$u = \phi\bar{\phi} + \phi'\bar{\phi}' + \dots + \phi^{n-1}\bar{\phi}^{n-1}$

$u' = \phi\bar{\phi}' + \phi'\bar{\phi} + \phi''\bar{\phi} + \phi'\bar{\phi}'' + \ldots + \phi^{n}\bar{\phi}^{n-1} + \phi^{n-1}\bar{\phi}^{n}$

$\therefore |u'| \leq |\phi||\phi'| + |\phi'||\phi| + |\phi''||\phi'| + \ldots + |\phi^{n}||\phi^{n-1}| + |\phi^{n-1}||\phi^{n}|$

$|u'| \leq 2|\phi||\phi'| + 2|\phi'||\phi''| + \ldots + 2|\phi^{n}||\phi^{n-1}|$ —— ①

ϕ satisfy the eq^n $L(y) = 0$

Since, ϕ satisfy the eq^n $L(y) = 0$

$L(\phi) = \phi^{n} + a_1(x)\phi^{n-1} + \ldots + a_n(x)\phi = 0$

$\Rightarrow \phi^{n} = -a_1(x)\phi^{n-1} - \ldots - a_n(x)\phi = 0$

$\Rightarrow |\phi^{n}| \leq |a_1(x)||\phi^{n-1}| + \ldots + |a_n(x)||\phi|$

$\Rightarrow |\phi^{n}| \leq b_1|\phi^{n-1}| + b_2|\phi^{n-2}| + \ldots + b_n|\phi|$

$(\because |a_j(x)| \leq b_j \;,\; j = 1, 2, \ldots, n)$

put these value in eq^n ①

$|u'| \leq 2|\phi||\phi'| + 2|\phi'||\phi''| + \ldots + 2|\phi^{n-1}|[b_1|\phi^{n-1}| + b_2|\phi^{n-2}| + \ldots b_n|\phi|]$

$|u'| \leq 2|\phi||\phi'| + 2|\phi'||\phi''| + \ldots + 2b_1|\phi^{n-1}|^2 + 2b_2|\phi^{n-2}||\phi^{n-1}| + \ldots + 2b_n|\phi||\phi^{n-1}|$

$2|b||c| \leq |b|^2 + |c|^2$

$|u'| \leq [|\phi|^2 + |\phi'|^2] + [|\phi'|^2 + |\phi''|^2] + \ldots + 2b_1|\phi^{n-1}|^2 + b_2[|\phi^{n-2}|^2 + |\phi^{n-1}|^2] + \ldots + b_n[|\phi|^2 + |\phi^{n-1}|^2]$

$|u'| \leq (1+b_n)|\phi|^2 + (2+b_{n-1})|\phi'|^2 + \ldots + |\phi^{n-1}|^2[1 + 2b_1 + b_2 + \ldots + b_n]$

Now, each bracketed term in RHS is less than or equal to $2K$.

$|u'| \leq 2K(|\phi|^2 + |\phi'|^2 + \ldots + |\phi^{n-1}|^2)$

$|u'| \leq 2K\|\phi\|^2$ — from ⊛

$|u'| \leq 2Ku$

$\Rightarrow -2Ku \leq u' \leq 2Ku$ —— ②

First we consider

$u' \leq 2Ku$

Multiplying by e^{-2Kx}, we get

$u'e^{-2Kx} \leq 2Ku\,e^{-2Kx}$

$\Rightarrow u'e^{-2Kx} - 2Ku\,e^{-2Kx} \leq 0$

$\Rightarrow (u \cdot e^{-2Kx})' \leq 0$ —— ③

Let $x_0 \in I$, $x > x_0$ we integrate

$[u \cdot e^{-2Kx}]_{x_0}^{x} \leq 0$

$u(x)e^{-2Kx} - u(x_0)e^{-2Kx_0} \leq 0$

$u(x)e^{-2Kx} \leq u(x_0)e^{-2Kx_0}$ —— ④

$u(x) \leq \dfrac{u(x_0)e^{-2Kx_0}}{e^{-2Kx}}$

$\Rightarrow u(x) \leq u(x_0)e^{-2Kx_0} \cdot e^{2Kx}$

$\Rightarrow u(x) \leq u(x_0)e^{-2K(x_0 - x)}$ $\forall x \geq x_0$

$\Rightarrow \|\phi(x)\|^2 \leq \|\phi(x_0)\|^2 e^{2K(x-x_0)}$

squaring both side

$\|\phi(x)\| \leq \|\phi(x_0)\| e^{K(x-x_0)}$ —— ⑤

Similarly,

By consider other inequality

$-2Ku \leq u'$

$\|\phi(x)\| \geq \|\phi(x_0)\| \cdot e^{-K(x-x_0)}$

$\Rightarrow \|\phi(x_0)\| e^{-K(x-x_0)} \leq \|\phi(x)\|$ —— ⑥

from eq^n ⑤ & ⑥

$\|\phi(x_0)\| e^{-K(x-x_0)} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{K(x-x_0)}$ —— ⑦

for $x \leq x_0$

Similary, find from eq^n ⑦

$\|\phi(x_0)\| e^{-K(x_0-x)} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{K(x_0-x)}$ —— ⑧

$(\because x \leq x_0)$

Combing eq^n ⑦ & ⑧

$\|\phi(x_0)\| e^{-K|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{K|x-x_0|}$, $\forall x \in I$

Hence, the proof.

* Uniqueness Theorem :-

Let x_0 be any point in I. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be any n constant. then their is a atmost one solution ϕ of $L(y)=0$ on I satisfy $\phi(x_0)=\alpha_1$, $\phi'(x_0)=\alpha_2$, $\dots$, $\phi^{n-1}(x_0)=\alpha_n$

Proof :-

Consider the initial value problem $L(y)=0$

i.e. $L(y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)y = 0$

with initial condition $y(x_0) = \alpha_1$, $y'(x_0)=\alpha_2, \dots$ $y^{n-1}(x_0) = \alpha_n$

where, $a_1, a_2, \dots, a_n$ are continuous function on I containing point x_0

Let ϕ & ψ be solution of above initial value problem.

Define a function χ on I as

$\chi = \phi - \psi$

Claim, $\chi(x) = 0$, $\forall x \in I$

Let x be any point in I other than x_0 & let J be the close bounded interval on I containing x_0 & x on these interval J continuous $a_j(x)$, $j=1,2,\dots,n$ are bounded (since continuous function define on an close bounded interval is bounded)

i.e. a non-negative const. b_j, $j=1,2,\dots,n$

$|a_j(x)| < b_j$

Now, the function χ define on J such that

$L(\chi) = L(\phi) - L(\psi)$

$= 0 - 0$

$L(\chi) = 0$

$\chi(x_0) = \phi(x_0) - \psi(x_0)$

$= \alpha_1 - \alpha_1$

$= 0$

$\chi'(x_0) = \phi'(x_0) - \psi'(x_0)$

$= \alpha_2 - \alpha_2$

$= 0$

$\chi''(x_0) = \phi''(x_0) - \psi''(x_0)$

$= \alpha_3 - \alpha_3$

$= 0$

$\vdots$

$\chi^n(x_0) = \phi^{n-1}(x_0) - \psi^{n-1}(x_0)$

$= \alpha_n - \alpha_n$

$= 0$

but we know that χ is a solution of $L(y)=0$ on some interval J containing the point x_0.

$\|\chi(x_0)\| e^{-k|x-x_0|} \le \|\chi(x)\| \le \|\chi(x_0)\| e^{+k|x-x_0|}$

$\|\chi(x_0)\|^2 = |\chi(x_0)|^2 + |\chi'(x_0)|^2 + \dots + |\chi^{n-1}(x_0)|^2$

$= 0 + 0 + \dots + 0$

$\|\chi(x_0)\|^2 = 0$

$\|\chi(x_0)\| = 0$

$0 \le \|\chi(x)\| \le 0$

Thus, the given initial value problem has atmost one solution ϕ containg point x_0.

$\|\chi(x)\| = 0$

$\chi(x) = 0$

$\phi(x) - \psi(x) = 0$

$\phi(x) = \psi(x)$.

* **The solution of the homogeneous eqn :-**

Let n function $\phi_1, \phi_2, \dots, \phi_n$ define on interval I are said to be linearly independant iff there exist $c_1, c_2, \dots, c_n$ such that

$c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x) = 0$

where, $c_1 = c_2 = c_3 = \dots = c_n = 0$

* **Theorem :-**

Let $\phi_1, \phi_2, \dots, \phi_n$ be any n linearly independant solution of $L(y)=0$ on I. If ϕ is any solution of $L(y)=0$ on I then show that & it can be represented by

$\phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x)$.

OR

i.e. Any set of n linearly independant solution of $L(y)=0$ on I is the basis for solution of $L(y)=0$ on I.
(L.I & linear span)

Proof :-

Let $L(y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x) y = 0$ —— ①

where, $a_1, a_2, \dots, a_n$ are continuous functio on I.

Let $\phi_1, \phi_2, \dots, \phi_n$ be n linearly independant solution of $L(y)=0$ on an interval I.

i.e. $W(\phi_1, \phi_2, \dots, \phi_n)(x) \neq 0, \ \forall x \in I$ —— ②

Let x_0 be any fix point in I & ϕ be any solution of $L(y)=0$ satisfying the condition

$\phi(x_0) = \alpha_1, \ \phi'(x_0) = \alpha_2, \dots, \phi^{n-1}(x_0) = \alpha_n$ —— ③

where, $\alpha_1, \alpha_2, \dots, \alpha_n$ are constant.

Since, $W(\phi_1, \phi_2, \dots, \phi_n)(x) \neq 0, \ \forall x \in I$

$\therefore W(\phi_1, \phi_2, \dots, \phi_n)(x_0) \neq 0$, for some $x_0 \in I$

There exist system of non-homogeneous eqn.

$$\left.\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) + \dots + c_n\phi_n(x_0) &= \phi(x_0) = \alpha_1 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \dots + c_n\phi_n'(x_0) &= \phi'(x_0) = \alpha_2 \\ &\vdots \\ c_1\phi_1^{n-1}(x_0) + c_2\phi_2^{n-1}(x_0) + \dots + c_n\phi_n^{n-1}(x_0) &= \phi^{n-1}(x_0) = \alpha_n \end{aligned}\right\} ④$$

where, $c_1, c_2, \dots, c_n$ not all zero simultaneously satisfying eqn ④.

For this choise of $c_1, c_2, \dots, c_n$

Define a new function ψ.

$\psi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x)$ —— ⑤

$L(\psi) = c_1 L(\phi_1) + c_2 L(\phi_2) + \dots + c_n L(\phi_n)$

$= 0$ ($\because \phi_1, \phi_2, \dots, \phi_n$ are soln of $L(y)=0$)

$L(\psi) = 0$ —— ⑥

$\psi(x_0) = c_1\phi_1(x_0) + c_2\phi_2(x_0) + \dots + c_n\phi_n(x_0)$

$\psi(x_0) = \alpha_1$

$\vdots$

$\psi^{n-1}(x_0) = c_1\phi_1^{n-1}(x_0) + c_2\phi_2^{n-1}(x_0) + \dots + c_n\phi_n^{n-1}(x_0)$

$\psi^{n-1}(x_0) = \alpha_n$

$\therefore \psi(x_0) = \alpha_1, \ \psi'(x_0) = \alpha_2, \dots, \psi^{n-1}(x_0) = \alpha_n$ —— ⑦

From eqn ⑥ & ⑦, we get

$\psi(x)$ is a solution of initial value problem

$L(y) = 0$

$y(x_0) = \alpha_1, \ y'(x_0) = \alpha_2, \dots, y^{n-1}(x_0) = \alpha_n$

but from uniqueness theorem

$\psi(x) = \phi(x)$ ($\because$ by eqn ③)

$\therefore \phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x)$ is a solution of $L(y)=0$ on I.

Hence, the proof.

Wronskian :-

Let $\phi_1, \phi_2, \dots, \phi_n$ be n function on an interval I then the determinant

$$\begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \vdots \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix}$$

is called wronskain of function & it is denoted by $W(\phi_1, \phi_2, \dots, \phi_n)(x)$.

* **Theorem :-**

Let $\phi_1, \phi_2, \dots, \phi_n$ be n solution of $L(y)=0$ on an interval I then they are L.I. iff $W(\phi_1, \phi_2, \dots, \phi_n)(x) \neq 0$, $\forall x \in I$.

Proof :-

Let $\phi_1, \phi_2, \dots, \phi_n$ be n solution of $L(y)=0$ on an interval I. —①

~~Let $\phi_1, \phi_2, \dots, \phi_n$ be~~

Assume that $W(\phi_1, \phi_2, \dots, \phi_n)(x) \neq 0$ —②

claim, we proof that $\phi_1, \phi_2, \dots, \phi_n$ are linearly independant. on I.

$c_1, c_2, \dots, c_n$ are constant

$$c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x) = 0, \ \forall x \in I \quad \text{—③}$$

Diff. w.r.t. x $(n-1)$ times of eqⁿ ③, we get

$$\left.\begin{aligned} & c_1\phi_1'(x) + c_2\phi_2'(x) + \dots + c_n\phi_n'(x) = 0 \\ & c_1\phi_1''(x) + c_2\phi_2''(x) + \dots + c_n\phi_n''(x) = 0 \\ & \vdots \\ & c_1\phi_1^{n-1}(x) + c_2\phi_2^{n-1}(x) + \dots + c_n\phi_n^{n-1}(x) = 0 \end{aligned}\right\} ④$$

for some x_0, we have, $x_0 \in I$

$$\left.\begin{aligned} & c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \dots + c_n\phi_n'(x_0) = 0 \\ & c_1\phi_1''(x_0) + c_2\phi_2''(x_0) + \dots + c_n\phi_n''(x_0) = 0 \\ & \vdots \\ & c_1\phi_1^{n-1}(x_0) + c_2\phi_2^{n-1}(x_0) + \dots + c_n\phi_n^{n-1}(x_0) = 0 \end{aligned}\right\} ⑤$$

This is the system of n homogeneous equation in n unknowns $c_1, c_2, \dots, c_n$

The determinant of coefficient of eqⁿ ⑤ is just the $W(\phi_1, \phi_2, \dots, \phi_n)(x_0)$ which is different from zero.

i.e. $W(\phi_1, \phi_2, \dots, \phi_n)(x_0) \neq 0$

$\Rightarrow$ The system of eqⁿ ⑤ has unique solution is given by $c_1 = c_2 = \dots = c_n = 0$

This shows that $\phi_1, \phi_2, \dots, \phi_n$ are linearly independant on I.

conversely,

Assume that $\phi_1, \phi_2, \dots, \phi_n$ are linearly independant on I.

claim, to prove that $W(\phi_1, \phi_2, \dots, \phi_n)(x) \neq 0$ $\forall x \in I$

Assume that there exist a point x_0 in I such that $W(\phi_1, \phi_2, \dots, \phi_n)(x_0) = 0$, $x_0 \in I$

$$\begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) & \dots & \phi_n(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) & \dots & \phi_n'(x_0) \\ \vdots & & & \\ \phi_1^{n-1}(x_0) & \phi_2^{n-1}(x_0) & \dots & \phi_n^{n-1}(x_0) \end{vmatrix} = 0$$

$$\left.\begin{aligned} c_1\phi_1(x_0) + c_2\phi_2(x_0) + \dots + c_n\phi_n(x_0) &= 0 \\ c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \dots + c_n\phi_n'(x_0) &= 0 \\ &\vdots \\ c_1\phi_1^{n-1}(x_0) + c_2\phi_2^{n-1}(x_0) + \dots + c_n\phi_n^{n-1}(x_0) &= 0 \end{aligned}\right\} \text{⑥}$$

which is n constant $c_1, c_2, \dots, c_n$ such that not all constant are zero. Simultaneously, for this choice of $c_1, c_2, \dots, c_n$

Define a new function

$$\psi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x) \quad \text{—— ⑦}$$

$$\begin{aligned} L(\psi(x)) &= L(c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x)) \\ &= c_1 L(\phi_1(x)) + c_2 L(\phi_2(x)) + \dots + c_n L(\phi_n(x)) \\ &= 0 + 0 + \dots + 0 \qquad (\because \phi_1, \phi_2, \dots, \phi_n \text{ are sol}^n \text{ of } L(y)=0) \\ &= 0 \end{aligned}$$

$$\left.\begin{aligned} \psi(x_0) &= c_1\phi_1(x_0) + c_2\phi_2(x_0) + \dots + c_n\phi_n(x_0) = 0 \\ \psi'(x_0) &= c_1\phi_1'(x_0) + c_2\phi_2'(x_0) + \dots + c_n\phi_n'(x_0) = 0 \\ &\vdots \\ \psi^{n-1}(x_0) &= c_1\phi_1^{n-1}(x_0) + c_2\phi_2^{n-1}(x_0) + \dots + c_n\phi_n^{n-1}(x_0) = 0 \end{aligned}\right\} \text{by ⑥}$$

$$\therefore \psi(x_0) = \psi'(x_0) = \dots = \psi^{n-1}(x_0) = 0 \quad \text{—— ⊛}$$

we know,

$$\|\psi(x_0)\| e^{-k|x-x_0|} \leq \|\psi(x)\| \leq \|\psi(x_0)\| e^{k|x-x_0|} \quad \text{—— ⑧}$$

where, $\|\psi(x_0)\| = [\,|\psi(x_0)|^2 + |\psi'(x_0)|^2 + \dots + |\psi^{n-1}(x_0)|^2]^{1/2}$

$\|\psi(x_0)\| = 0 \qquad \because$ by ⊛

putting in eq^n ⑧, we get

$$0 \leq \|\psi(x)\| \leq 0$$

$\Rightarrow \|\psi(x)\| = 0$

$\Rightarrow \psi(x) = 0, \quad \forall x \in I$

$\Rightarrow c_1\phi_1(x) + c_2\phi_2(x) + \dots + c_n\phi_n(x) = 0$

where, $c_1, c_2, \dots, c_n$ are not zero simultaneously.

$\therefore \phi_1, \phi_2, \dots, \phi_n$ are linearly dependant on I.

This is contradiction to our assumption that $\phi_1, \phi_2, \dots, \phi_n$ are linearly independant.

This arieses because of our wrong assumption $W(\phi_1, \phi_2, \dots, \phi_n)(x_0) = 0$

Hence, $W(\phi_1, \phi_2, \dots, \phi_n)(x) \neq 0, \ \forall x \in I$

Hence, the proof.

IMP * **Theorem :-**

Let $\phi_1, \phi_2, \dots, \phi_n$ be n solution of $L(y)=0$ on an interval I & let x_0 be any point in I. Then show that

$$W(\phi_1, \phi_2, \dots, \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] W(\phi_1, \phi_2, \dots, \phi_n)(x_0)$$

Proof :-

Let $L(y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x) y = 0$

where, $a_1(x), a_2(x), \dots, a_n(x)$ are continuous function on I.

Let $\phi_1, \phi_2, \dots, \phi_n$ be n solution of $L(y)=0$ on I.

Now,

$$W(x) = W(\phi_1, \phi_2, \dots, \phi_n)(x)$$

$$W(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \vdots \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix} \quad \text{—— *}$$

By the well known property of determinant we know W' is the sum of the n determinants of the form.

$W'(x) = V_1 + V_2 + \dots + V_k + \dots + V_n$

where, V_k is order of $n \times n$ & it differ from k^{th} row

The k^{th} row of V_k is obtain by differentiating the k^{th} row of W.

Thus, we have

$$W'(x) = \begin{vmatrix} \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix} + \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1''(x) & \phi_2''(x) & \dots & \phi_n''(x) \\ \vdots & & & \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix}$$

$$+ \dots + \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix} + \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n' \\ \vdots & & & \\ \phi_1^{n-2}(x) & \phi_2^{n-2}(x) & \dots & \phi_n^{n-2} \\ \phi_1^{n}(x) & \phi_2^{n}(x) & \dots & \phi_n^{n} \end{vmatrix} \quad \text{— (1)}$$

we see that first $n-1$ determinant $V_1, V_2, \dots V_{n-1}$ on RHS of eq^n (1) are zero as each of them two identically row. ~~is~~ hence.

$$W'(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \vdots \\ \phi_1^{n}(x) & \phi_2^{n}(x) & \dots & \phi_n^{n}(x) \end{vmatrix} \quad \text{— (2)}$$

Since, $\phi_i \ (i = 1, 2, \dots, n)$ is a solution of $L(y) = 0$.

i.e. $\phi_i^n + a_1(x)\phi_i^{n-1} + \dots + a_n(x)\phi_i = 0 \quad \forall i = 1, 2, \dots, n$

$\phi_i^n = -a_1(x)\phi_i^{n-1} - \dots - a_n(x)\phi_i$

$\phi_i^n = -(a_1(x)\phi_i^{n-1} + \dots + a_n(x)\phi_i)$

$\phi_i^n = -\sum_{j=0}^{n-1} a_{n-j}\phi_i^j(x) \quad , \forall j = 1, \dots, n$

Hence, the eq^n (2) becomes

$$W'(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \vdots \\ -\sum_{j=0}^{n-1} a_{n-j}\phi_1^j(x) & -\sum_{j=0}^{n-1} a_{n-j}\phi_2^j(x) & \dots & -\sum_{j=0}^{n-1} a_{n-j}\phi_n^j(x) \end{vmatrix}$$

We know the value of determinate is unchange if we multiply number & add it.

Thus we multiply by a_n, second row by a_{n-1} & $(n-1)^{th}$ row by a_2 & we get,

Then

$$W'(x) = \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & \dots & \phi_n' \\ \vdots & & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2} & \dots & \phi_n^{n-2} \\ -a_1\phi_1^{n-1} & -a_1\phi_2^{n-1} & \dots & -a_1\phi_n^{n-1} \end{vmatrix}$$

$$W'(x) = -a \begin{vmatrix} \phi_1 & \phi_2 & \dots & \phi_n \\ \phi_1' & \phi_2' & & \phi_n' \\ \vdots & & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2} & \dots & \phi_n^{n-2} \\ \phi_1^{n-1} & \phi_2^{n-1} & \dots & \phi_n^{n-1} \end{vmatrix}$$

$W'(x) = -a_1 W(x) \quad$ — by (*)

$W'(x) + a_1(x) W(x) = 0$

($a_1, a_2, \dots, a_n$ are continuous fu^n)

$$\frac{W'(x)}{W(x)} = -a_1(x)$$

Integrating to $x \to x_0$

$$\int_{x_0}^{x} \frac{W'(x)}{W(x)} dx = -\int_{x_0}^{x} a_1(t)\, dt$$

$$[\log W(x)]_{x_0}^{x} = -\int_{x_0}^{x} a_1(t)\, dt$$

$$\log W(x) - \log(x_0) = -\int_{x_0}^{x} a_1(t)\, dt$$

$$\log\left[\frac{W(x)}{W(x_0)}\right] = -\int_{x_0}^{x} a_1(t)\, dt$$

$$\frac{W(x)}{W(x_0)} = \exp\left[-\int_{x_0}^{x} a_1(t)\, dt\right]$$

$$W(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\, dt\right] W(x_0) \quad \text{—④}$$

$$W(\phi_1, \phi_2, \dots, \phi_n)(x) = \exp\left[-\int_{x_0}^{x} a_1(t)\, dt\right] W(\phi_1, \phi_2, \dots, \phi_n)(x_0) \quad \text{—③}$$

Hence, the proof.

~~Theorem~~ :-

* Corollary :-

If the coefficient a_i of ~~x~~ for all $i = 1, 2, \dots, n$ are constant. Then the eq^n ③ gives

$$W(x) = e^{-a_1(x - x_0)} W(x_0)$$

i.e. $W(\phi_1, \phi_2, \dots, \phi_n)(x) = e^{-a_1(x - x_0)} W(\phi_1, \phi_2, \dots, \phi_n)(x_0)$

Proof :-

Let $L(y) = y^n + a_1(x) y^{n-1} + \dots + a_n(x) y = 0$

where, $a_1, a_2, \dots, a_n$ are constants.

Let $\phi_1, \phi_2, \dots, \phi_n$ be n solution of $L(y) = 0$ on I.

Now,

$$W(x) = W(\phi_1, \phi_2, \dots, \phi_n)(x)$$

$$W(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & & \phi_n'(x) \\ \vdots & & & \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix}$$

By the well known property of determinant we know W' is the sum of the n determinant of the form

$$W'(x) = V_1 + V_2 + \dots + V_k + \dots + V_n$$

where, V_k is order of $n \times n$ & it differ from k^{th} row.

The k^{th} row of V_k is obtained by differentiating the k^{th} row of W.

Thus, we have

$$W'(x) = \begin{vmatrix} \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix} + \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1''(x) & \phi_2''(x) & \dots & \phi_n''(x) \\ \vdots & & & \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix}$$

$$+ \dots + \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & \dots & \phi_n^{n-1}(x) \end{vmatrix} + \begin{vmatrix} \phi_1(x) & \phi_2(x) & \dots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \dots & \phi_n'(x) \\ \vdots & & & \\ \phi_1^{n-2}(x) & \phi_2^{n-2}(x) & \dots & \phi_n^{n-2}(x) \\ \phi_1^{n}(x) & \phi_2^{n}(x) & \dots & \phi_n^{n}(x) \end{vmatrix} \quad \text{—①}$$

we see that first $n-1$ determinant $V_1, V_2, \dots, V_{n-1}$ on RHS of eq^n ① are zero as each of them two identically row. Hence

$$W'(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \cdots & \phi_n'(x) \\ \vdots & & & \\ \phi_1^n(x) & \phi_2^n(x) & \cdots & \phi_n^n(x) \end{vmatrix} \quad \text{—— (2)}$$

since, ϕ_i $(i=1,2,\ldots,n)$ is a solution of $L(y)=0$

i.e. $\phi_i^n + a_1(x)\phi_i^{n-1} + \ldots + a_n(x)\phi_i = 0 \quad \forall\, i=1,2,\ldots,n$

$\phi_i^n = -a_1(x)\phi_i^{n-1} - \ldots - a_n(x)\phi_i$

$\phi_i^n = -(a_1(x)\phi_i^{n-1} + \ldots + a_n(x)\phi_i)$

$= -\sum_{j=0}^{n-1} a_{n-j}\phi_i^j(x) \quad , \quad \forall\, j=1,2,\ldots,n$

Hence, eq^n (2) becomes

$$W'(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & & \phi_n'(x) \\ \vdots & & & \vdots \\ -\sum_{j=0}^{n-1} a_{n-j}\phi_1^j(x) & -\sum_{j=0}^{n-1} a_{n-j}\phi_2^j(x) & \cdots & -\sum_{j=0}^{n-1} a_{n-j}\phi_n^j(x) \end{vmatrix}$$

We know the value of determinate is unchange if we multiply number & add it.

Thus, we multiply by a_n 1^{st} row, 2^{nd} row by a_{n-1} & $(n-1)^{th}$ row by a_2 & we get

Then.

$$W'(x) = \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n \\ \phi_1'(x) & \phi_2'(x) & \cdots & \phi_n' \\ \vdots & & & \vdots \\ \phi_1^{n-2} & \phi_2^{n-2}(x) & \cdots & \phi_n^{n-2}(x) \\ -a_1\phi_1^{n-1}(x) & -a_1\phi_2^{n-1}(x) & \cdots & -a_1\phi_n^{n-1}(x) \end{vmatrix}$$

$$W'(x) = -a_1 \begin{vmatrix} \phi_1(x) & \phi_2(x) & \cdots & \phi_n(x) \\ \phi_1'(x) & \phi_2'(x) & \cdots & \phi_n'(x) \\ \vdots & & & \vdots \\ \phi_1^{n-2}(x) & \phi_2^{n-2}(x) & & \phi_n^{n-2}(x) \\ \phi_1^{n-1}(x) & \phi_2^{n-1}(x) & & \phi_n^{n-1}(x) \end{vmatrix}$$

$W'(x) = -a_1 W(x)$

$W'(x) + a_1(x)W(x) = 0$

$(a_1, a_2, \ldots, a_n$ are constant.)

$\frac{W'(x)}{W(x)} = -a_1(x)$

Integrating from $x \to x_0$

$\int_{x_0}^{x} \frac{W'(x)}{W(x)}\,dx = -a_1 \int_{x_0}^{x} dt$

$\left[\log W(x)\right]_{x_0}^{x} = -a_1(x-x_0)$

$\log W(x) - \log W(x_0) = -a_1(x-x_0)$

$\log\left[\frac{W(x)}{W(x_0)}\right] = -a_1(x-x_0)$

$\frac{W(x)}{W(x_0)} = e^{-a_1(x-x_0)}$

$W(x) = e^{-a_1(x-x_0)}\,W(x_0)$

$W(\phi_1,\phi_2,\ldots,\phi_n)(x) = e^{-a_1(x-x_0)}\,W(\phi_1,\phi_2,\ldots,\phi_n)(x_0)$

Hence, the proof.

* Theorem :-

The non-homogeneous eqⁿ of order second.

<u>Statement</u> :- Consider the eqⁿ

$$L(y) = y'' + a_1(x)y' + a_2(x)y = b(x)$$

where, $a_1(x), a_2(x)$ & $b(x)$ are continuous funⁿ on an interval I. Show that every solⁿ ψ of $L(y) = b(x)$ on I can be written as

$$\psi = c_1\phi_1 + c_2\phi_2 + \psi_p$$

where, ψ_p is particular solution of $L(y) = b(x)$; ϕ_1, ϕ_2 are two linearly independant solution of $L(y) = 0$ & c_1, c_2 constant.

A particular solution is given by

$$\psi_p = \int_{x_0}^{x} \left[\frac{\phi_1(t)\phi_2(x) - \phi_1(x)\phi_2(t)}{W(\phi_1, \phi_2)(t)} \right] b(t)\, dt$$

Proof :-

Let ψ & ψ_p be the two solution of $L(y) = b(x)$

and $L(y) = y'' + a_1y' + a_2y = 0$

Then, $L(\psi - \psi_p) = L(\psi) - L(\psi_p)$

$= b(x) - b(x)$

$L(\psi - \psi_p) = 0$

This shows that the function $(\psi - \psi_p)$ be the solution of $L(y) = 0$

We know that, ϕ_1, ϕ_2 are two linearly independant solution of $L(y) = 0$.

Then every solution of $L(y)$ has of the form $c_1\phi_1 + c_2\phi_2$, where, c_1, c_2 are constant.

$\psi - \psi_p = c_1\phi_1 + c_2\phi_2$

clearly, a function $c_1\phi_1 + c_2\phi_2$ cannot be a solution of $L(y) = b(x)$ unless $b(x) = 0$ on I

Suppose that $\phi(x) = u_1\phi_1(x) + u_2\phi_2(x)$ be the solution of $L(y) = b(x)$ on interval I. Then,

$$y'' + a_1y' + a_2y = b$$

$\Rightarrow \phi'' + a_1\phi' + a_2\phi = b$

$\Rightarrow \phi = u_1\phi_1 + u_2\phi_2$

$\Rightarrow \phi' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2$

$\Rightarrow \phi'' = u_1''\phi_1 + u_1'\phi_1' + u_1'\phi_1' + u_1\phi_1'' + u_2''\phi_2 + u_2'\phi_2' + u_2'\phi_2' + u_2\phi_2''$

$\phi'' = 2(u_1'\phi_1' + u_2'\phi_2') + u_1''\phi_1 + u_1\phi_1'' + u_2''\phi_2 + u_2\phi_2''$

$\therefore \phi'' + a_1\phi' + a_2\phi = b$

$2(u_1'\phi_1' + u_2'\phi_2') + u_1''\phi_1 + u_1\phi_1'' + u_2''\phi_2 + u_2\phi_2'' + a_1(u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2) + a_2(u_1\phi_1 + u_2\phi_2) = b$

$u_1(\phi_1'' + a_1\phi_1' + a_2\phi_1) + u_2(\phi_2'' + a_1\phi_2' + a_2\phi_2) + 2(u_1'\phi_1' + u_2'\phi_2') + u_1''\phi_1 + u_2''\phi_2 + a_1\phi_1u_1' + a_1u_2'\phi_2 = b$

$u_1(L(\phi_1)) + u_2(L(\phi_2)) + 2(u_1'\phi_1' + u_2'\phi_2') + u_1''\phi_1 + u_2''\phi_2 + a_1(u_1'\phi_1 + u_2'\phi_2) = b$

$\Rightarrow 2(u_1'\phi_1' + u_2'\phi_2') + u_1''\phi_1 + u_2''\phi_2 + a_1(u_1'\phi_1 + u_2'\phi_2) = b$ —— ①

$\{\because L(\phi_1) = 0, L(\phi_2) = 0\}$

If $u_1'\phi_1 + u_2'\phi_2 = 0$ —— ②

then $(u_1'\phi_1 + u_2'\phi_2)' = 0$

$\phi_1'u_1' + \phi_1u_1'' + \phi_2'u_2' + \phi_2u_2'' = 0$

$(\phi_1u_1'' + \phi_2u_2'') + (\phi_1'u_1' + \phi_2'u_2') = 0$

$\phi_1u_1'' + \phi_2u_2'' = -(\phi_1'u_1' + \phi_2'u_2')$ —— (*).

Then eqⁿ ① become

$\Rightarrow 2(u_1'\phi_1' + u_2'\phi_2') - (\phi_1'u_1' + \phi_2'u_2') + a_1(0) = b$ $(\because$ by ② & *$)$

$u_1'\phi_1' + u_2'\phi_2' = b$ —— ③

$\therefore$ eqⁿ ② & ③ is the non-homogeneous system of equation u_1' & u_2' ~~then~~

Thus, if we can find two function $u_1(x)$ & $u_2(x)$ such that

$\phi_1u_1' + \phi_2u_2' = 0$

$\phi_1'u_1' + \phi_2'u_2' = b$

Then, $u_1\phi_1 + u_2\phi_2$ be the solution of $L(y) = b(x)$ on the solving above two eqⁿ for u_1' & u_2'

$$u_1' = \frac{-b\phi_2}{W(\phi_1,\phi_2)}$$
$$u_2' = \frac{b\phi_1}{W(\phi_1,\phi_2)}$$
} by Cramer's rule

Integrating both eq^n bet^n the limit x_0 to x, we get

$$u_1'(x) = \frac{-b(x)\phi_2(x)}{W(\phi_1,\phi_2)(x)}$$

$$[u_1(x)]_{x_0}^{x} = \int_{x_0}^{x} \frac{-b(t)\cdot\phi_2(t)}{W(\phi_1,\phi_2)(t)}\,dt$$

$$u_1(x) - u_1(x_0) = \int_{x_0}^{x} \frac{-b(t)\cdot\phi_2(t)}{W(\phi_1,\phi_2)(t)}\,dt$$

$$u_1(x) = u_1(x_0) + \int_{x_0}^{x} \frac{-b(t)\phi_2(t)}{W(\phi_1,\phi_2)(t)}\,dt$$

$$u_2'(x) = \frac{b(x)\cdot\phi_1(x)}{W(\phi_1,\phi_2)(x)}$$

$$[u_2(x)]_{x_0}^{x} = \int_{x_0}^{x} \frac{b(t)\,\phi_1(t)}{W(\phi_1,\phi_2)(t)}\,dt$$

$$u_2(x) = u_2(x_0) + \int_{x_0}^{x} \frac{b(t)\phi_1(t)}{W(\phi_1,\phi_2)(t)}\,dt$$

The solution $\phi(x) = u_1(x)\phi_1(x) + u_2(x)\phi_2(x)$

$$\phi(x) = \left[u_1(x_0) + \int_{x_0}^{x} \frac{-b(t)\phi_2(t)}{W(\phi_1,\phi_2)(t)}\,dt\right]\phi_1(x) + \left[u_2(x_0) + \int_{x_0}^{x} \frac{b(t)\phi_1(t)}{W(\phi_1,\phi_2)(t)}\,dt\right]\phi_2(x)$$

$$\therefore\ \phi(x) = u_1(x_0)\phi_1(x) + u_2(x_0)\phi_2(x) + \int_{x_0}^{x} \frac{\phi_1(t)\phi_2(x) - \phi_2(t)\phi_1(x)}{W(\phi_1,\phi_2)(t)}\,b(t)\,dt$$

$$\Rightarrow\ \phi(x) = c_1\phi_1(x) + c_2\phi_2(x) + \psi_p$$

where, $c_1 = u_1(x_0)$ & $c_2 = u_2(x_0)$ &

$$\psi_p = \int_{x_0}^{x} \frac{\phi_1(t)\phi_2(x) - \phi_2(t)\phi_1(x)}{W(\phi_1,\phi_2)(t)}\,b(t)\,dt$$

Thus, $\phi = c_1\phi_1 + c_2\phi_2 + \psi_p$ be the solution of $L(y) = b(x)$.

Non-homogeneous eq^n with variable & order n :-

Theorem :-

Let b be a continuous function on an interval I & let $\phi_1, \phi_2, \dots, \phi_n$ be n basis for linearly independent solution of $L(y) = 0$ on I. Every solution ψ of $L(y) = b(x)$ can be written as

$$\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$$

where, $\psi_p \to$ particular solution of $L(y) = b(x)$ & $c_1, c_2, \dots, c_n$ are n constant. Every such of ψ is a solution of $L(y) = b(x)$. A particular solution $\psi_p(x)$ is given by

$$\psi_p(x) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(t)\cdot b(t)}{W(\phi_1,\phi_2,\dots,\phi_n)(t)}\,dt$$

where, $W_k(t)$ is the determinant obtain from $W(\phi_1, \phi_2, \dots, \phi_n)$ by replacing k^{th} column i.e. $[\phi_k, \phi_k', \dots, \phi_k^{n-1}]$ by $[0, 0, \dots, 1]$.

~~Proof~~ :-

Proof :-

~~Theorem~~ :-

Let $L(y) = y^n + a_1(x)y^{n-1} + \dots + a_n(x)y = b(x)$ ... ①

where $a_1(x), a_2(x)$ & $b(x)$ are continuous function on an interval I.

Suppose ψ & ψ_p be a solution of $L(y) = b(x)$ & ψ be any other sol^n of $L(y) = b(x)$

$$L(\psi - \psi_p) = L(\psi) - L(\psi_p)$$
$$= b(x) - b(x)$$
$$= 0$$

Then, $\psi - \psi_p$ is a solution of $L(y) = 0$.

but we know that any sol^n of $L(y) = 0$ can be written as

$$\psi - \psi_p = c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$$

where $c_1, c_2, \dots, c_n$ are constant. &

$\phi_1, \phi_2, \dots, \phi_n$ are n-linearly independant solution of $L(y) = 0$ on I. $\{\because \phi_1, \phi_2, \dots, \phi_n$ be basis for sol^n of $L(y) = 0$

Thus, $\psi = \psi_p + c_1\phi_1 + c_2\phi_2 + \dots + c_n\phi_n$ —— (2)

To find ψ_p, we use variation of constant method i.e. we try to find the functions $u_1, u_2, \dots, u_n$ s.t.

$$\psi_p = \sum_{k=1}^{n} u_k \phi_k(x)$$

$$\psi_p = u_1\phi_1(x) + u_2\phi_2(x) + \dots + u_n\phi_n(x) \quad \text{—— (3)}$$

is a solution of $L(y) = b(x)$

First we proof that ψ_p is a solution of $L(y) = b(x)$. i.e. we claim that

$$(\psi_p)^n + a_1(x)(\psi_p)^{n-1} + \dots + a_n(x)\psi_p = b(x) \quad \text{—— (4)}$$

Differentiating eq^n (3) w.r.t. x

$$(\psi_p)' = u_1\phi_1' + u_1'\phi_1 + u_2\phi_2' + u_2'\phi_2 + \dots + u_n\phi_n' + u_n'\phi_n$$
$$= (u_1\phi_1' + u_2\phi_2' + \dots + u_n\phi_n') + (u_1'\phi_1 + u_2'\phi_2 + \dots + u_n'\phi_n) \quad \text{—— (5)}$$

we choose $u_1, u_2, \dots, u_n$ such that

$$u_1'\phi_1 + u_2'\phi_2 + \dots + u_n'\phi_n = 0 \quad \text{—— (6)}$$

Then, from eq^n (5)

$$(\psi_p)' = u_1\phi_1' + u_2\phi_2' + \dots + u_n\phi_n' \quad \text{—— (7)}$$

Differentiating eq^n (7) w.r.t. x

$$(\psi_p)'' = u_1\phi_1'' + u_1'\phi_1' + u_2\phi_2'' + u_2'\phi_2' + \dots + u_n\phi_n'' + u_n'\phi_n'$$
$$= (u_1\phi_1'' + u_2\phi_2'' + \dots + u_n\phi_n'') + (u_1'\phi_1' + u_2'\phi_2' + \dots + u_n'\phi_n')$$

If we choose $u_1, u_2, \dots, u_n$ such that

$$u_1'\phi_1' + u_2'\phi_2' + \dots + u_n'\phi_n' = 0 \quad \text{—— (8)}$$

Then,

$$(\psi_p)'' = u_1\phi_1'' + u_2\phi_2'' + \dots + u_n\phi_n'' \quad \text{—— (9)}$$

Continuous this process we obtain in general,

$$(\psi_p)^n = (u_1\phi_1^n + u_2\phi_2^n + \dots + u_n\phi_n^n) + (u_1'\phi_1^{n-1} + u_2'\phi_2^{n-1} + \dots + u_n'\phi_n^{n-1}) \quad \text{—— (10)}$$

If we choose

$$u_1'\phi_1^{n-1} + u_2'\phi_2^{n-1} + \dots + u_n'\phi_n^{n-1} = b(x) \quad \text{—— (11)}$$

Then, eq^n (10) becmes.

$$(\psi_p)^n = u_1\phi_1^n + u_2\phi_2^n + \dots + u_n\phi_n^n + b(x) \quad \text{—— (12)}$$

Thus, we have $u_1', u_2', \dots, u_n'$ satisfing the eq^n

$$\left.\begin{aligned} u_1'\phi_1 + u_2'\phi_2 + \dots + u_n'\phi_n &= 0 \\ u_1'\phi_1' + u_2'\phi_2' + \dots + u_n'\phi_n' &= 0 \\ &\vdots \\ u_1'\phi_1^{n-2} + u_2'\phi_2^{n-2} + \dots + u_n'\phi_n^{n-2} &= 0 \\ u_1'\phi_1^{n-1} + u_2'\phi_2^{n-1} + \dots + u_n'\phi_n^{n-1} &= b(x) \end{aligned}\right\} \text{(13)}$$

we get

$$\left.\begin{aligned} &\text{v} \quad \psi_p = u_1\phi_1 + u_2\phi_2 + \dots + u_n\phi_n \\ &\text{iv} \quad \psi_p' = u_1\phi_1' + u_2\phi_2' + \dots + u_n\phi_n' \\ &\text{iii} \quad \psi_p'' = u_1\phi_1'' + u_2\phi_2'' + \dots + u_n\phi_n'' \\ &\quad \vdots \\ &\text{ii} \quad \psi_p^{n-1} = u_1\phi_1^{n-1} + u_2\phi_2^{n-1} + \dots + u_n\phi_n^{n-1} \\ &\text{i} \quad \psi_p^n = u_1\phi_1^n + u_2\phi_2^n + \dots + u_n\phi_n^n + b(x) \end{aligned}\right\} \text{(14)}$$

Multiply eq^n (i) by 1, eq^n (ii) by $a_1, \dots,$ eq^n (ii) by a_{n-2}, eq^n (iv) by a_{n-1} & eq^n (v) by a_n & adding we get

$$\psi_p^n + a_1\psi_p^{n-1} + a_2\psi_p^{n-2} + \dots + a_{n-2}\psi_p'' + a_{n-1}\psi_p' + a_n\psi_p$$
$$= u_1(\phi_1^n + a_1\phi_1^{n-1} + \dots + a_n\phi_1) + u_2(\phi_2^n + a_1\phi_2^{n-1} + \dots + a_n\phi_2) + \dots + u_n(\phi_n^n + a_1\phi_n^{n-1} + \dots + a_n\phi_n) + b(x)$$
$$= u_1(L(\phi_1)) + u_2(L(\phi_2)) + \dots + u_n(L(\phi_n)) + b(x)$$

($\because \phi_1, \phi_2, \dots, \phi_n$ are L.I. sol^n of $L(y)=0$)

$$= b(x)$$

$$\therefore \psi_p^n + a_1\psi_p^{n-1} + \dots + a_n\psi_p = b(x)$$

$\therefore \psi_p = u_1\phi_1 + u_2\phi_2 + \dots + u_n(\phi_n)$ is the solution of $L(y) = b(x)$

where, $u_1, u_2, \dots, u_n$ are the determine
To find $u_1, u_2, \dots, u_n$ we solve the set of non-homogeneous eqⁿ.
Thus, we have

$$\begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \phi_1' & \phi_2' & \cdots & \phi_n' \\ \vdots & & & \\ \phi_1^{n-1} & \phi_2^{n-1} & \cdots & \phi_n^{n-1} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b(x) \end{bmatrix} \quad (\because \text{by } ⑬)$$

By Cramer's rule.

$$u_k'(x) = \frac{\Delta_k(x)}{W(\phi_1, \phi_2, \dots, \phi_n)(x)} \quad \text{—— } ⑮ \quad (\because \phi_1, \phi_2, \dots, \phi_n \text{ are L.I. soln of } L(y)=0)$$

(Hence, $W(\phi_1, \phi_2, \dots, \phi_n)(x) \neq 0$)

where,
$\Delta_k(x)$ is a determinant obtain from W by replacing k^{th} column.

$$\Delta_k(x) = [\phi_k, \phi_k', \dots, \phi_k^{n-1}]\,[0, 0, \dots, b]$$

$$\Delta_k(x) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_k & \cdots & \phi_n \\ \phi_1' & \phi_2' & & \phi_k' & & \phi_n' \\ \vdots & & & & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \cdots & \phi_k^{n-1} & & \phi_n^{n-1} \end{vmatrix}$$

$$\Delta_k(x) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & 0 & \cdots & \phi_n \\ \phi_1' & \phi_2' & & 0 & & \phi_n' \\ \vdots & & & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & & b & \cdots & \phi_n^{n-1} \end{vmatrix}$$

$$\Delta_k(x) = b \begin{vmatrix} \phi_1 & \phi_2 & \cdots & 0 & \cdots & \phi_n \\ \phi_1' & \phi_2' & & 0 & & \phi_n' \\ \vdots & \vdots & & \vdots & & \vdots \\ \phi_1^{n-1} & \phi_2^{n-1} & \cdots & 1 & & \phi_n^{n-1} \end{vmatrix}$$

$$\Delta_k(x) = b\, W_k(\phi_1, \phi_2, \dots, \phi_n)(x)$$

where, $W_k(\phi_1, \phi_2, \dots, \phi_n)(x)$ is the determinant obtained from $W(\phi_1, \phi_2, \dots, \phi_n)(x)$. By replacing k^{th} column of $W(0, 0, \dots, 1)$ then we have, $u_1', u_2', \dots, u_n'$ satisfy the eqⁿ.

$$u_k' = \frac{W_k(\phi_1, \phi_2, \dots, \phi_n)(x)\, b(x)}{W(\phi_1, \phi_2, \dots, \phi_n)(x)} \quad \text{—— } ⑯ \quad (\because \text{by } ⑮)$$

Integrating x_0 to x, we get

If x_0 be any point

$$u_k(x) = u_k(x_0) + \int_{x_0}^{x} \frac{W_k(\phi_1, \phi_2, \dots, \phi_n)(t)\, b(t)}{W(\phi_1, \phi_2, \dots, \phi_n)(t)}\, dt$$

$$u_k(x) = \int_{x_0}^{x} \frac{W_k(\phi_1, \phi_2, \dots, \phi_n)(t)\, b(t)}{W(\phi_1, \phi_2, \dots, \phi_n)(t)}\, dt$$

$$\psi_p = \sum_{k=1}^{n} \phi_k(x)\, u_k(x)$$

$$\psi_p = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{W_k(\phi_1, \phi_2, \dots, \phi_n)(t)\, b(t)}{W(\phi_1, \phi_2, \dots, \phi_n)(t)}\, dt$$

Hence, the proof.

IMP * 10M

Reduction of order :-

Statement :- Let ϕ_1 be a solution of $L(y)=0$ on an interval I & suppose that $\phi_1(x) \neq 0$ on I.

If $v_2, v_3, \ldots, v_n$ is any basis on I for the solution of linear equation of order $(n-1)$.

$\phi_1 v^{n-1} + \ldots + (n\phi_1^{(n-1)} + (n-1)a_1\phi_1^{(n-2)} + \ldots + a_{n-1}\phi_1)v = 0$

& if $v_k = u_k'$, $k = 2, 3, \ldots, n$. Then, show that $\phi_1, u_2\phi_1, u_3\phi_1, \ldots, u_n\phi_1$ is a basis for the solution of $L(y)=0$ on I.

Solution :-

Let $L(y) = y^n + a_1(x)y^{n-1} + \ldots + a_n(x)y = 0$ —①

Let ϕ_1 be the solution of $L(y)=0$ on I.

We try to find a solution ϕ of the form

$\phi(x) = u(x)\cdot\phi_1(x)$

where, u is a some function.

if $\phi(x) = u(x)\cdot\phi_1(x)$ is a solution of $L(y)=0$

i.e. $L(\phi) = 0$

$\Rightarrow L(u\phi_1) = 0$

$\Rightarrow (u\phi_1)^n + a_1(x)(u\phi_1)^{n-1} + \ldots + a_n(x)(u\phi_1) = 0$

$\Rightarrow [u^n\phi_1 + u^{n-1}\phi_1' n + \ldots + \phi_1^n u] + a_1(x)[u^{n-1}\phi_1 + u^{n-2}\phi_1'(n-1) + \ldots + \phi_1^{n-1}u] + \ldots + a_n(x)(u\phi_1) = 0$

$\Rightarrow u^n\phi_1 + u^{n-1}[n\phi_1' + \phi_1 a_1(x)]$
$+ u^{n-2}[\frac{n(n-1)}{2}\phi_1'' + a_1(x)\cdot(n-1)\phi_1' + a_2(x)\phi_1]$
$+ \ldots + u'[n\phi_1^{n-1} + a_1(x)(n-1)\phi_1^{n-2} + \ldots + a_{n-1}(x)\phi_1]$
$+ u[\phi_1^n + a_1(x)\phi_1^{n-1} + \ldots + a_n(x)\phi_1] = 0$

$\because \phi_1$ is a solution of $L(y)=0$

$\therefore L(\phi_1) = 0$ i.e. $\phi_1^n + a_1(x)\phi_1^{n-1} + \ldots + a_n(x)\phi_1 = 0$

$L(u\phi_1) = u^n\phi_1 + u^{n-1}[n\phi_1' + \phi_1 a_1(x)]$
$+ u^{n-2}[\frac{n(n-1)}{2}\phi_1'' + a_1(x)(n-1)\phi_1' + a_2(x)\phi_1]$
$+ \ldots + u'[n\phi_1^{n-1} + a_1(x)(n-1)\phi_1^{n-2} + \ldots + a_{n-1}(x)\phi_1] = 0$ —②

Define $v = u'$ —③

i.e. $v(x) = u'(x)$

$u(x) = \int_{x_0}^{x} v(t)\,dt$ —④

from eqⁿ ②,

$v^{n-1}\phi_1 + v^{n-2}[n\phi_1' + a_1(x)\phi_1]$
$+ v^{n-3}[\frac{n(n-1)}{2}\phi_1'' + a_1(x)(n-1)\phi_1' + a_2(x)\phi_1]$
$+ \ldots + v[n\phi_1^{n-1} + a_1(x)(n-1)\phi_1^{n-2} + \ldots + a_{n-1}(x)\phi_1] = 0$ —⑤

This shows that the function v satisfies the eqⁿ $M(y)=0$, where

$M(y) = b_1 y^{n-1} + b_2 y^{n-2} + b_3 y^{n-3} + \ldots + b_n y = 0$

where, $b_1 = \phi_1$

$b_2 = n\phi_1' + a_1(x)\phi_1$

$b_3 = \frac{n(n-1)}{2}\phi_1'' + a_1(x)(n-1)\phi_1' + a_2(x)\phi_1$

$\vdots$

$b_n = n\phi_1^{n-1} + a_1(x)(n-1)\phi_1^{n-2} + \ldots + a_{n-1}(x)\phi_1$

Since, $b_1 = \phi_1(x) \neq 0$

Hence, eqⁿ ⑤ being a linear differential eqⁿ of order $(n-1)$ & it has $(n-1)$ linearly independant solution $v_2, v_3, \ldots, v_n$ on I. Corresponding value of $u(x)$ are obtained as

$u_k(x) = \int_{x_0}^{x} v_k(t)\,dt$, $k = 2, 3, \ldots, n$ —⑥

but then by assumption $u_k(x)\phi_1(x)$ is a solution of $L(y)=0$ on I

Thus, the function $\phi_1, u_2\phi_1, u_3\phi_1, \ldots, u_n\phi_1$ are solution of $L(y)=0$ on I.

Now, we will proof that these solution are linearly independant on I.

$c_1, c_2, \ldots, c_n$ be constant such that

$$c_1\phi_1 + c_2u_2\phi_1 + \dots + c_nu_n\phi_1 = 0$$

$$\Rightarrow \phi_1[c_1 + c_2u_2 + \dots + c_nu_n] = 0$$

$$\Rightarrow c_1 + c_2u_2 + \dots + c_nu_n = 0 \quad \because (\phi_1 \neq 0) \quad \text{---} (*)$$

Differentiated, we get

$$\Rightarrow c_2u_2' + c_3u_3' + \dots + c_nu_n' = 0$$

but $u_k' = v_k$

$$\therefore c_2v_2 + c_3v_3 + \dots + c_nv_n = 0$$

but $v_2, v_3, \dots, v_n$ is a basis on I.

$$\therefore c_2 = c_3 = c_4 = \dots = c_n = 0$$

put these value in (*)

$$\Rightarrow c_1 = 0$$

$$\therefore c_1 = c_2 = c_3 = \dots = c_n = 0$$

$\therefore \phi_1 + u_2\phi_1 + u_3\phi_1 + \dots + u_n\phi_1$ are the solution of $L(y) = 0$

$\therefore \phi_1, u_2\phi_1, u_3\phi_1, \dots, u_n\phi_1$ is a basis for the solution of $L(y) = 0$ on I.

Hence, the proof.

IMP/IOM * **Theorem :—**

If ϕ_1 is a solution of $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ on I & second solution $\phi_2(x)$ of $L(y) = 0$ is given by

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2}\, ds$$

The function ϕ_1, ϕ_2 form a basis of the solution of $L(y) = 0$ on I

Proof :—

Let $\phi_1(x) \neq 0$ be a solution of $L(y) = 0$

$$L(y) = y'' + a_1(x)y' + a_2(x)y = 0$$

$$\Rightarrow L(\phi_1) = \phi_1'' + a_1(x)\phi_1' + a_2(x)\phi_1 = 0 \quad \text{---} (1)$$

Now, we try to find solution of $L(y) = 0$ of the form $\phi = u\phi_1$, where u is a some function

If $\phi = u\phi_1$ is a solution of $L(y) = 0$ we must have $L(u\phi_1) = 0$.

$$(u\phi_1)'' + a_1(x)(u\phi_1)' + a_2(x)(u\phi_1) = 0$$

$$(u\phi_1)' = (u'\phi_1 + u\phi_1')$$

$$(u\phi_1)'' = (u''\phi_1 + u'\phi_1' + u'\phi_1' + u\phi_1'')$$

$$(u''\phi_1 + u'\phi_1' + u'\phi_1' + u\phi_1'') + a_1(x)(u'\phi_1 + u\phi_1') + a_2(x)(u\phi_1) = 0$$

$$u''\phi_1 + u'(2\phi_1' + a_1(x)\phi_1) + u(\phi_1'' + a_1(x)\phi_1' + a_2(x)\phi_1) = 0$$

but ϕ_1 is a solution of $L(y) = 0$

$$\therefore L(\phi_1) = 0$$

$$\therefore u''\phi_1 + u'[2\phi_1' + a_1(x)\phi_1] = 0 \quad (\because \text{by } (1))$$

Here, $u_k' = v_k$

$$\Rightarrow u' = v \quad \text{---} (*)$$

$$\therefore \phi_1v' + (2\phi_1' + a_1(x)\phi_1)v = 0 \quad \text{---} (2)$$

which is linear eqn of order 1.

$$\Rightarrow \phi_1v' + 2\phi_1'v + a_1(x)\phi_1v = 0$$

Multiply by ϕ_1, we get

$$\Rightarrow \phi_1^2v' + 2\phi_1\phi_1'v + a_1(x)\phi_1^2v = 0$$

$$\Rightarrow (v\phi_1^2)' + a_1(x)\phi_1^2v = 0$$

$$\Rightarrow \frac{(v\phi_1^2)'}{v\phi_1^2} = -a_1(x)$$

Integrating x_0 to x, we get

$$\int \frac{(v\phi_1^2)'}{v\phi_1^2} = -\int_{x_0}^{x} a_1(x)\,dx + \log c$$

$$\Rightarrow \log(v\phi_1^2) = -\int_{x_0}^{x} a_1(x)\,dx + \log c$$

$$\Rightarrow \log\left(\frac{v\phi_1^2}{c}\right) = -\int_{x_0}^{x} a_1(x)\,dx$$

$$\Rightarrow \frac{v\phi_1^2}{c} = \exp\left[-\int_{x_0}^{x} a_1(x)\,dx\right]$$

$$\Rightarrow v\phi_1^2 = c\cdot\exp\left[-\int_{x_0}^{x} a_1(x)\,dx\right]$$

where, c is constant of integration

$$\Rightarrow V(x) = \frac{c}{\phi_1^2(x)} \exp\left[-\int_{x_0}^{x} a_1(x)\,dx\right]$$

We know any constant multiply of the solution is also solution of eq^n ②,

∴ we take ~~V(x)/c~~

$$V(x) = \frac{1}{\phi_1^2(x)} \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right] \quad \text{——③}$$

From eq^n ③ & (*)

$$u'(x) = \frac{1}{\phi_1^2(x)} \exp\left[-\int_{x_0}^{x} a_1(t)\,dt\right]$$

Integrating, we get

$$u(x) = \int_{x_0}^{x} \frac{1}{\phi_1^2(s)} \exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right] ds$$

The second solution ϕ_2 is given by

$\phi_2 = u\phi_1$

Hence, the other solution $L(y)=0$ is given by

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2} ds$$

Now, we show that ϕ_1 & ϕ_2 are linearly independant.

Consider $c_1\phi_1 + c_2\phi_2 = 0$

$c_1\phi_1 + c_2(u\phi_1) = 0$

$\Rightarrow \phi_1(c_1 + uc_2) = 0$

$\Rightarrow c_1 + uc_2 = 0 \quad , \phi_1 \neq 0$

Differentiating, we get

$u'c_2 = 0$

$\Rightarrow c_2 V = 0$

$\Rightarrow c_2 = 0 \quad$ (V is a basis)

∴ $c_1 = 0$

∴ $c_1 = c_2 = 0$

∴ ϕ_1 & ϕ_2 are linearly independant on I.

Hence, the proof.

Examples

1) Consider the equation $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$ where, $a_1(x)$ & $a_2(x)$ are continuous function of x, on some interval I. Let $\phi_1, \phi_2, \psi_1, \psi_2$ be two basis of solution of $L(y)=0$. show that there exist a non-zero const. K s.t.

$W(\phi_1,\phi_2)(x) = K \cdot W(\psi_1,\psi_2)(x)$

Given $L(y) = y'' + a_1(x)y' + a_2(x)y = 0$

where $a_1(x)$ & $a_2(x)$ are continuous function of x on some interval I. ~~Let~~

Let $\phi_1, \phi_2, \psi_1, \psi_2$ be two basis of solution of $L(y)=0$

i.e. ϕ_1, ϕ_2 & ψ_1, ψ_2 are L.I. solution of $L(y)=0$

∴ $W(\phi_1,\phi_2)(x) \neq 0 \quad , \forall x \in I$

∴ $W(\phi_1,\phi_2)(x_0) \neq 0 \quad , x_0 \in I$

∴ $W(\phi_1,\phi_2)(x_0) = K_1$

Similarly

$W(\psi_1,\psi_2)(x) \neq 0 \quad , \forall x \in I$

∴ $W(\psi_1,\psi_2)(x_0) \neq 0 \quad , x_0 \in I$

∴ $W(\psi_1,\psi_2)(x_0) = K_2$

Also, we know that

$$W(\phi_1,\phi_2)(x) = e^{-\int_{x_0}^{x} a_1(t)dt}\, W(\phi_1,\phi_2)(x_0)$$

$$W(\psi_1,\psi_2)(x) = e^{-\int_{x_0}^{x} a_1(t)dt}\, W(\psi_1,\psi_2)(x_0)$$

Taking the ratio of the two eq^n, we get

$$\frac{W(\phi_1,\phi_2)(x)}{W(\psi_1,\psi_2)(x)} = \frac{W(\phi_1,\phi_2)(x_0)}{W(\psi_1,\psi_2)(x_0)} = \frac{K_1}{K_2}$$

$$= \frac{K_1}{K_2}$$

$$= K$$

$$\frac{W(\phi_1,\phi_2)(x)}{W(\psi_1,\psi_2)(x)} = k$$

$\therefore W(\phi_1,\phi_2)(x) = k\,W(\psi_1,\psi_2)(x)$

Hence, the proof

2] One solution of $L(y) = y'' + \frac{1}{4x^2}y = 0$, for $x>0$ is $\phi_1(x) = x^{1/2}$. Show that there is a another solution ψ of the form $\psi = u\phi_1$, where u is a some function

Given, $L(y) = y'' + \frac{1}{4x^2}y = 0$, for $x>0$

$L(\psi) = L(u\phi_1)$

$L(u\phi_1) = (u\phi_1)'' + \frac{1}{4x^2}(u\phi_1)$

$= (ux^{1/2})'' + \frac{1}{4x^2}(ux^{1/2})$

$(ux^{1/2})' = u'x^{1/2} + u\frac{1}{2}x^{-1/2}$

$(ux^{1/2})'' = u''x^{1/2} + u'\frac{1}{2}x^{-1/2} + u'\frac{1}{2}x^{-1/2} + u\left(\frac{-1}{4}\right)x^{-3/2} = u''x^{1/2} + u'x^{-1/2} - \frac{1}{4}ux^{-3/2}$

$L(u\phi_1) = L(u\cdot x^{1/2})'' + \frac{1}{4x^2}(ux^{1/2})$

$= u''x^{1/2} + u'x^{-1/2} - \frac{1}{4}u\cdot x^{-3/2} + \frac{1}{4x^2}(ux^{1/2})$

$= u''x^{1/2} + x^{-1/2}u' - \frac{1}{4}ux^{-3/2} + \frac{1}{4}ux^{-3/2}$

$= u''x^{1/2} + x^{-1/2}u'$

If ψ is a solution then $L(\psi)=0$

then $L(u\phi_1) = 0$

$\therefore x^{1/2}u'' + x^{-1/2}u' = 0$

$u'' + \frac{u'}{x} = 0$ $\quad$ divide $x^{1/2}$

If $u' = v$

$v' + \frac{v}{x} = 0$

$v' = -\frac{v}{x}$

$\frac{v'}{v} = \frac{-1}{x}$

Integrating

$\int \frac{v'}{v} = \int \frac{-1}{x}$

$\Rightarrow \log(v) = -\log x + \log c$

$\Rightarrow \log(v) + \log x = \log c$

$\Rightarrow \log(vx) = \log c$

$\Rightarrow vx = c$

$\Rightarrow v = \frac{c}{x}$

but $u' = v$

$\therefore u' = \frac{c}{x}$

Integrating

$u = c\log x$

$\therefore \psi = u\phi_1$

$\Rightarrow \psi = c\log x\cdot x^{1/2}$

$\Rightarrow \psi = (\log x\cdot x^{1/2})$

3] One solution of $x^2y'' - xy' + y = 0$, $x>0$ is $\phi_1(x) = x$. Find the another independant solution of eq^n.

Consider $x^2y'' - xy' + y = 0$, $x>0$ — ①

$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$

$\therefore a_1(x) = -\frac{1}{x}$, & $a_2(x) = \frac{1}{x^2}$

Since, $\phi_1(x) = x$ is a solution of eqⁿ ①.
If ϕ_2 is another independant solⁿ of the ~~another~~ given eqⁿ

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2}\, ds$$

$$= \cancel{x \int_1^x e^{-\int a_1(t)}}$$

$$= x \int_1^x \frac{e^{-\int_1^s -\frac{1}{t}dt}}{s^2}\, ds$$

$$= x \int_1^x \frac{e^{[\log t]_1^s}}{s^2}\, ds$$

$$= x \int_1^x \frac{e^{[\log s - \log 1]}}{s^2}\, ds$$

$$= x \int_1^x \frac{e^{\log s}}{s^2}\, ds$$

$$= x \int_1^x \frac{s}{s^2}\, ds$$

$$= x \int_1^x \frac{1}{s}\, ds$$

$$= x[\log s]_1^x$$

$$= x[\log x - \log 1]$$

$$= x \log x$$

$\therefore \phi_2(x) = x\log x$ is another independant solution of given eqⁿ ①.

4] One solution of $x^2y'' - 7xy' + 15y = 0,\ x>0$ is $\phi_1(x) = x^3$ Find second solⁿ

Consider $x^2y'' - 7xy' + 15y = 0$

$$y'' - \frac{7}{x}y' + 15\frac{y}{x^2} = 0$$

$\therefore a_1 = \frac{-7}{x}, \quad a_2(x) = \frac{15}{x^2}$

Since, $\phi_1(x) = x^3$ is a solution of eqⁿ ①
If ϕ_2 is another independant solution of the given eqⁿ.

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2}\, ds$$

$$= x^3 \int_1^x \frac{e^{-\int_1^s \frac{-7}{t}dt}}{(s^3)^2}\, ds$$

$$= x^3 \int_1^x \frac{e^{\int_1^s \frac{7}{t}dt}}{s^6}\, ds$$

$$= x^3 \int_1^x \frac{e^{[7\log t]_1^s}}{s^6}\, ds$$

$$= x^3 \int_1^x \frac{e^{(7\log s - 7\log 1)}}{s^6}\, ds$$

$$= x^3 \int_1^x \frac{e^{7\log s}}{s^6}\, ds$$

$$= x^3 \int_1^x \frac{e^{\log s^7}}{s^6}\, ds$$

$$= x^3 \int_1^x \frac{s^7}{s^6}\, ds$$

$$= x^3 \int_1^x s\, ds$$

$$= x^3 \left[\frac{s^2}{2}\right]_1^x$$

$$\phi_2(x) = x^3\left[\frac{x^2}{2} - \frac{1}{2}\right]$$

$$\phi_2(x) = \frac{x^5}{2} - \frac{x^3}{2}$$

$$\phi_2(x) = \frac{1}{2}(x^5 - x^3)$$

(i.e. $\phi_2(x) = \frac{-1}{2}x^3 + \frac{1}{2}x^5$

i.e. $c_1 = \frac{-1}{2}$, $c_2 = \frac{1}{2}$

i.e. $\phi_1(x) = x^3$, $\phi_2(x) = x^5$)

5) One solution of $y'' - 4xy' + (4x^2-2)y = 0$ if $\phi_1(x) = e^{x^2}$. Find another independant sol^n.

Consider $y'' - 4xy' + (4x^2-2)y = 0$ —①

$a_1 = -4x$, $a_2(x) = 4x^2 - 2$

Since $\phi_1(x) = e^{x^2}$ is a solution of eq^n ① if ϕ_2 is another independant sol^n of the given eq^n

$$\phi_2(x) = \phi_1(x)\int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2}\,ds$$

$$= e^{x^2}\int_1^x \frac{e^{-\int_1^s -4t\,dt}}{(e^{s^2})^2}\,ds$$

$$= e^{x^2}\int_1^x \frac{e^{\left[4\frac{t^2}{2}\right]_1^s}}{e^{2s^2}}\,ds$$

$$= e^{x^2}\int_1^x \frac{e^{[2t^2]_1^s}}{e^{2s^2}}$$

$$\phi_2(x) = e^{x^2}\int_1^x \frac{e^{(2s^2-2)}}{e^{2s^2}}\,ds$$

$$= e^{x^2}\int_1^x \frac{e^{2s^2}\cdot e^{-2}}{e^{2s^2}}\,ds$$

$$= e^{x^2}\int_1^x \frac{e^{-2}}{\cancel{e^{2s^2}}}\,ds$$

$$\cancel{= e^{x^2}\cdot e^{-2}\int_1^x e^{-2s^2}\,ds}$$

$$\cancel{= e^{x^2}\cdot e^{-2}\int e^{-2s^2}}$$

$$= e^{x^2}e^{-2}\int_1^x ds$$

$$= e^{x^2}e^{-2}[s]_1^x$$

$$= e^{x^2}e^{-2}(x-1)$$

$$\phi_2(x) = \frac{e^{x^2}}{e^2}(x-1)$$

6) One of solution $xy'' - (x+1)y' + y = 0$, $x > 0$ $\phi_1(x) = e^x$. Find another independant sol^n.

Consider $xy'' - (x+1)y' + y = 0$

$$y'' - \frac{(x+1)}{x}y' + \frac{y}{x} = 0$$

$$a_1 = -\frac{(x+1)}{x} = -1 - \frac{1}{x},\quad a_2 = \frac{1}{x}$$

Since $\phi_1(x) = e^x$ is a solution of eq^n if ϕ_2 is another independant sol^n of the given eq^n.

$$\phi_2(x) = \phi_1(x)\int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2}\,ds.$$

$$\phi_2(x) = e^x \int_1^x \frac{e^{-\int_1^s (-1-\frac{1}{t})dt}}{(e^s)^2} ds$$

$$= e^x \int_1^x \frac{e^{-[-t-\log t]_1^s}}{e^{2s}} ds$$

$$= e^x \int_1^x \frac{e^{[t+\log t]_1^s}}{e^{2s}} ds$$

$$= e^x \int_1^x \frac{e^{(s+\log s-1-\log 1)}}{e^{2s}} ds$$

$$= e^x \int_1^x \frac{e^{(s-1+\log(s-1))}}{e^{2s}} ds$$

$$= e^x \int_1^x \frac{e^{(s+\log s-1)}}{e^{2s}} ds$$

$$= e^x \int_1^x \frac{e^s \cdot e^{\log s} \cdot e^{-1}}{e^{2s}} ds$$

$$= e^x \int_1^x \frac{e^s \cdot s \cdot e^{-1}}{e^{2s}} ds$$

$$= e^x \int_1^x \frac{e^{(s-1)} \cdot s}{e^{2s}} ds \qquad e^x \int_1^x \frac{s \cdot e^s e^{-1}}{e^{2s}} ds$$

$$= e^x \left[s \cdot \int e^{-s-1} - \int \frac{d}{ds} s \cdot \int e^{-s-1} ds \right] \quad e^{-1} e^x \int_1^x \frac{s}{e^s} ds$$

$$= e^x \left[s \cdot \frac{e^{-s-1}}{-s-1} - \int 1 \cdot \frac{e^{-s-1}}{-s-1} ds \right] \quad e^{-1} e^x \int_1^x s \cdot e^{-s} ds$$

$$= e^x \left[\frac{s}{-s-1} e^{-s-1} - \frac{1}{-s-1} \cdot \frac{e^{-s-1}}{-s-1} \right]$$

$$-e^{-1}(x+1) + 2e^{-2}e^x \qquad s \int \frac{e^{-s}}{-1}$$

$$= e^x \left[\frac{s}{-s-1} e^{-s-1} - \cancel{\frac{1}{-2s-2} e^{-s-1}} \right]$$

$$= e^x \left[\cancel{\left(\frac{1}{-s-1} - \frac{1}{-2s-2} \right)} e^{-s-1} \right]$$

$$\phi_2(x) = e^{-1} e^x \left[s \cdot \frac{e^{-s}}{-1} - \int_1^x \left[\frac{e^{-s}}{-1} \right] ds \right]$$

$$= e^{-1} e^x \left[-s \cdot e^{-s} + \frac{e^{-s}}{-1} \right]_1^x$$

$$e^{-1} e^x \left[s \cdot \frac{e^{-s}}{-1} + \frac{e^{-s}}{-1} \right]_1^x = e^{-1} e^x \left[-s \cdot e^{-s} - e^{-s} \right]_1^x$$

$$= e^{-1} e^x \left[-x \cdot e^{-x} - e^{-x} - (-1 \cdot e^{-1} - e^{-1}) \right]$$

$$= e^{-1} e^x \left[-x \cdot e^{-x} - e^{-x} + e^{-1} + e^{-1} \right]$$

$$= e^{x-1} \left[-x \cdot e^{-x} - e^{-x} + 2e^{-1} \right]$$

$$= e^{x-1} \left(e^{-x}(-1-x) + 2 \cdot e^{-1} \right)$$

$$= e^{x-1} \left[e^{-x}(-1-x) + 2 \cdot e^{-1} e^{x-1} \right]$$

$$= e^{-1}(-1-x) + 2 \cdot e^{x-2}$$

$$= e^{-1}(-1-x) + 2 e^{x-2}$$

$$= -e^{-1}(1+x) + 2 \cdot e^x \cdot e^{-2}$$

$$= 2 \cdot e^{-2} \cdot e^x - e^{-1}(1+x)$$

(i.e. $\phi_2(x) = 2 \cdot e^{-2} \cdot e^x - e^{-1}(1+x)$

i.e. $c_1 = 2 \cdot e^{-2}$ & $c_2 = e^{-1}$

i.e. $\phi_1(x) = e^x$ & $\phi_2(x) = 1+x$).

$\phi_2(x) = -e^{-1}(x+1) + 2 \cdot e^{-2} e^x$

$$\phi(x) = c_1 \phi_1(x) + c_2 \phi_2(x)$$

$$= c_1 e^x + c_2 \left[-(x+1)e^{-1} + 2e^{-2}e^x \right]$$

$$= (c_1 + 2e^{-2}c_2)e^x + (c_2 e^{-1})(-(x+1))$$

$$= d_1 e^x + d_2(-(x+1)).$$

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} e^x & -(x+1) \\ e^x & -1 \end{vmatrix}$$

$$= -e^x + e^x(x+1) = -e^x + x e^x + e^x$$

$$= x e^x$$

$$\neq 0$$

ϕ_1, ϕ_2 are L.I.

Q) Solve $y'' - \frac{2}{x^2} y = 0, \quad 0 < x < \infty$

$\phi_1(x) = x^2$. Find all soln of $y'' - \frac{2}{x^2} y = x$

Consider $y'' - \frac{2}{x^2} y = 0$

$a_1 = 0$.

Since, $\phi_1(x) = x^2$ is a solution of eqn
if ϕ_2 is another independant soln of the
given eqn

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2}\, ds$$

$$= x^2 \int_1^x \frac{e^{-\int_1^s 0\,dt}}{(s^2)^2}\, ds$$

$$= x^2 \int_1^x \frac{e^0}{s^4}\, ds$$

$$= x^2 \int_1^x \frac{1}{s^4}\, ds$$

$$= x^2 \int_1^x s^{-4}\, ds$$

$$= x^2 \left[\frac{s^{-3}}{-3}\right]_1^x$$

$$= x^2 \left[\frac{1}{-3}\left(x^{-3} - x^{-1}\right)\right]$$

$$= x^2 \frac{1}{-3}\left(\frac{1}{x^3} - \frac{1}{x}\right)$$

$$= -\frac{x^2}{3x^3} + \frac{x^2}{3x}$$

$$= -\frac{1}{3x} + \frac{x}{3} = \frac{1}{3}\left(-\frac{1}{x} + x\right)$$

$$\phi_2(x) = x^2 \left[\frac{-1}{3}\left(x^{-3} - 1\right)\right]$$

$$= -\frac{1}{3}\frac{x^2}{x^3} + \frac{1}{3}x^2$$

$$= -\frac{1}{3x} + \frac{1}{3}x^2 = \frac{1}{3}\left(x^2 - \frac{1}{x}\right)$$

$\phi_1(x) = x^2, \quad \phi_2(x) = \frac{1}{x}$

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} x^2 & 1/x \\ 2x & \frac{-\log x}{x} \end{vmatrix} \quad \begin{vmatrix} x^2 & +1/x \\ 2x & \frac{-1}{x^2} \end{vmatrix}$$

$$= x^2 \log x \cdot \frac{-1}{x} - 2x \cdot \frac{1}{x}$$

$$= x^2 \log x - 2$$

$$= -1 - 2$$

$$= -3$$

$$\neq 0$$

$$u_1(x) = \int \frac{-b(x)\,\phi_2(x)}{W(\phi_1, \phi_2)}\, dx$$

$$= \int \frac{-x \cdot \frac{1}{x}}{-3}\, dx$$

$$= \int \frac{-1}{-3}\, dx$$

$$= +\frac{1}{3}x$$

$$u_1(x) = \frac{+1}{3}x$$

$$u_2(x) = -\int \frac{b(x)\,\phi_1(x)}{W(\phi_1,\phi_2)(x)}$$

$$= \int \frac{x \cdot x^2}{-3}\,dx$$

$$= \int \frac{x^3}{-3}\,dx$$

$$= \frac{-x^4}{12}.$$

$$\therefore \psi_p = u_1\phi_1 + u_2\phi_2$$

$$= \frac{+x}{3}x^2 - \frac{x^4}{12}\frac{1}{x}$$

$$= \frac{1}{3}x^3 - \frac{1}{12}x^3$$

$$= x^3\left(\frac{1}{3} - \frac{1}{12}\right)$$

$$= x^3\left(\frac{3}{12}\right)$$

$$= x^3\left(\frac{1}{4}\right)$$

$$= \frac{x^3}{4}$$

8) Solve $x^2y'' - 2y = 0$, $0 < x < \infty$
$\phi_1(x) = x^2$. Find all solution of
$x^2y'' - 2y = 2x - 1$

consider $x^2y'' - 2y = 0$

$a_1 = 0$

since, $\phi_1(x) = x^2$ is a solution of eq^n.
if ϕ_2 is another independant sol^n of the given eq^n.

$$\phi_2(x) = \phi_1(x) \int_{x_0}^{x} \frac{\exp\left[-\int_{x_0}^{s} a_1(t)\,dt\right]}{[\phi_1(s)]^2}\,ds$$

$$= x^2 \int_1^x \frac{e^{-\int_1^s 0\,dt}}{(s^2)^2}\,ds$$

$$= x^2 \int_1^x \frac{e^0}{s^4}\,ds$$

$$= x^2 \int_1^x \frac{1}{s^4}\,ds$$

$$= x^2 \int_1^x s^{-4}\,ds$$

$$= x^2 \left[\frac{s^{-3}}{-3}\right]_1^x$$

$$= x^2\left[\frac{-1}{3}(x^{-3} - 1)\right]$$

$$= x^2\left(\frac{-1}{3x^3} + \frac{1}{3}\right)$$

$$= \frac{-1}{3x} + \frac{x^2}{3}$$

$$= \frac{1}{3}\left(x^2 - \frac{1}{x}\right)$$

$\phi_1(x) = x^2, \quad \phi_2(x) = 1/x$

$$W(\phi_1,\phi_2)(x) = \begin{vmatrix} x^2 & 1/x \\ 2x & -1/x^2 \end{vmatrix}$$

$$= x^2\left(\frac{-1}{x^2}\right) - 2x\left(\frac{1}{x}\right)$$

$$= -1-2$$

$$= -3$$

$$\neq 0$$

$$u_2(x) = \int \frac{+b(x)\,\phi_1(x)}{W(\phi_1,\phi_2)(x)}\,dx$$

$$= \int \frac{-(2x-1)\,1/x}{-3}\,dx$$

$$= \frac{1}{3}\int (2x-1)\frac{1}{x}\,dx$$

$$= \frac{1}{3}\int 2-\frac{1}{x}\,dx$$

$$= \frac{1}{3}\left[2x - \log x\right]$$

$$= \frac{2}{3}x - \frac{\log x}{3}$$

$$= +\int \frac{\frac{2x-1}{x^2}\cdot x^2}{-3}\,dx$$

$$= -\int \frac{2x-1}{3}\,dx$$

$$= -\frac{1}{3}\int 2x-1\,dx$$

$$= -\frac{1}{3}\left[2\frac{x^2}{2} - x\right]$$

$$= -\frac{1}{3}\left[x^2 - x\right]$$

$$u_1(x) = \int \frac{b(x)\,\phi_2(x)}{W(\phi_1,\phi_2)(x)}\,dx$$

$$= \int \frac{-(2x-1)/x^2 \cdot x^2}{-3}\,dx$$

$$= +\frac{1}{3}\int \frac{(2x-1)\,x^2}{x^3}\,dx \qquad = +\frac{1}{3}\int \frac{2}{x^2} - \frac{1}{x^3}$$

$$= \frac{-1}{3}\int 2x^2 - x^3\,dx \qquad = \frac{+1}{3}\left[2\cdot(-2)x^{-3} - (-3)x^{-4}\right]$$

$$= \frac{-1}{3}\left[\frac{2x^4}{4} - \frac{x^3}{3}\right] = \frac{+1}{3}\left[\frac{-4}{x^3} + \frac{3}{x^4}\right]$$

$$u_1 = -\frac{1}{3}\left[2\cdot\frac{x^{-1}}{-1} - \frac{x^{-2}}{-2}\right]$$

$$= -\frac{1}{3}\left[\frac{-2}{x} + \frac{1}{2x^2}\right]$$

$$u_1 = -\frac{2}{3x} + \frac{1}{6x^2}$$

$$u_2(x) = -\frac{1}{3}\left[\frac{x^4}{2} - \frac{x^3}{3}\right] \qquad u_1 = \frac{-4}{3x^3} + \frac{1}{x^4}$$

$$= \frac{-x^4}{6} + \frac{x^3}{9}$$

$$\psi_p = u_1\phi_1 + u_2\phi_2$$

$$= \left(\frac{2}{3}x - \frac{\log x}{3}\right)x^2 + \left(\frac{x^3}{9} - \frac{x^4}{6}\right)\frac{1}{x}$$

$$= \frac{2}{3}x^3 - \frac{\log x\cdot x^2}{3} + \frac{x^2}{9} - \frac{x^3}{6}$$

$$= x^3\left(\frac{2}{3} - \frac{1}{6}\right) + \frac{x^2}{9} - \frac{x^2}{3}\log x^2$$

$$= \frac{1}{2}x^3 + \frac{x^2}{9} - \frac{x^2}{3}\log x^2$$

$$u_1 = -\frac{2}{3x} + \frac{1}{6x^2}$$

$$u_2 = -\frac{1}{3}(x^2 - x)$$

$$\psi_p = u_1\phi_1 + u_2\phi_2$$

$$= \left(-\frac{2}{3x} + \frac{1}{6x^2}\right)x^2 + \left(\frac{-1}{3}(x^2-x)\right)\frac{1}{x}$$

$$= -\frac{2x}{3} + \frac{1}{6} - \frac{1}{3}x + \frac{1}{3}$$

$$= x\left(-\frac{2}{3} - \frac{1}{3}\right) + \left(\frac{1}{3} + \frac{1}{6}\right)$$

$$= \frac{x}{3}/ + 1/6 \quad -x + \frac{1}{2}$$

$$\boxed{\psi_p = \frac{1}{2} - x}$$

9) One solution $L(y) = x^3y''' - 3x^2y'' + 6xy' - 6y = 0$
$\phi_1(x) = x$. Find basis for solution of $L(y)=0$, for $x > 0$.

$L(y) = x^3y''' - 3x^2y'' + 6xy' - 6y = 0$ —— ①

$\phi_1(x) = x$ ~~be is the~~ soln of eqn ①

we try to find solution of eqn ① of the form

$\phi = u\phi_1$

$\phi = ux$

$L(u\phi_1) = L(ux) = 0$

$x^3(ux)''' - 3x^2(ux)'' + 6x(ux)' - 6(ux) = 0$

$(ux)' = u\cdot x' + u'x$ $\quad u'x + u$

$(ux)'' = u\cdot x'' + u'x' + u'x' + u''x = ux'' + 2u'x' + u''x$

$(ux)''' = u\cdot x''' + u'x'' + u'x'' + u'x'' + u'x'' + u''x' + u''x' + u'''x$

$(ux)''' = u\cdot x''' + 3u'x'' + 3u''x' + u'''x$ $\quad u'''x + u'' + u''+u''$

$x^3(u\cdot x''' + 3u'x'' + 3u''x' + u'''x) - 3x^2(ux'' + 2u'x' + u''x)$
$+ 6x(ux' + u'x) - 6(ux) = 0$

$x^3ux''' + x^3 3u'x'' + x^3 3u''x' + x^3u'''x - 3x^2ux'' - 6x^2u'x'$
$- 3x^2u''x + 6xux' + 6u'x - 6ux = 0$

$ux^5 + 3u'x^5 + 3u''x^4 + u'''x^4 - 3ux^4 - 6u'x^3$
$- 3u''x^3 + 6ux^2 + 6u'x - 6ux = 0$

$x^3(u'''x + 3u'') - 3x^2(u''x + 2u') + 6x(u'x + u) - 6ux = 0$

$x^4u''' + 3u''x^3 - 3u''x^3 - 6u'x^2 + 6u'x^2 + 6ux - 6ux = 0$

$x^4u''' = 0$

$\therefore u''' = 0$

putting $u' = v$

$u'' = v'$

$u''' = v''$

$\Rightarrow v'' = 0$

The characteristic polynomial is

$r^2 = 0$

$r = 0, 0$

$\phi_1(x) = 1, \quad \phi_2(x) = x$

$v = c_1v_1 + c_2v_2$

$= c_1\cdot 1 + c_2x$

$= c_1 + xc_2$

$v_1 = 1, \quad v_2 = x$

but we know that $u' = v$

$\therefore u_1' = 1, \quad u_2' = x,$

$\Rightarrow$ Integrating

$\therefore u_1 = x, \quad u_2 = \frac{x^2}{2}$

$\therefore$ basis forms of $L(y) = \{\phi_1, u_1\phi_1, u_2\phi_1\}$

$L(y) = \{x, x^2, x^3\}$

IMP 10) One solution $x^2y'' - xy' + y = 0$, $x > 0$
is $\phi_1(x) = x$

Find the soln ψ of $x^2y'' - xy' + y = x^2$
satisfying $\psi(1) = 1$, $\psi'(1) = 0$.

$\phi_2(x) = x\log x$
$u_1 = x - x\log x$
$u_2 = x$
$\psi_p = x^2$
$w = x$
$c_1 = ..., c_2 = -2$

Consider $x^2y'' - xy' + y = 0$

$\therefore y'' - \frac{1}{x}y' + \frac{1}{x^2}y = 0$

$a_1 = \frac{-1}{x}$

Since, $\phi_1(x) = x$ is a solution of eqn ①
if ϕ_2 is another independant soln of the given eqn

$$\phi_2(x) = \phi_1(x)\int_{x_0}^{x} \frac{\exp[-\int_{x_0}^{s} a_1(t)dt]}{[\phi_1(s)]^2}\,ds$$

$$\phi_2(x) = x \int_1^x \frac{e^{-\int_1^s \frac{1}{t} dt}}{s^2} ds$$

$$= x \int_1^x \frac{e^{[\log t]_1^s}}{s^2} ds$$

$$= x \int_1^x \frac{e^{(\log s - \log 1)}}{s^2} ds$$

$$= x \int_1^x \frac{e^{\log s}}{s^2} ds$$

$$= x \int_1^x \frac{s}{s^2} ds$$

$$= x \int_1^x \frac{1}{s} ds$$

$$= x [\log s]_1^x$$

$$= x (\log x - \log 1)$$

$$= x \log x$$

$\therefore \phi_1(x) = x, \quad \phi_2(x) = x \log x$

$$W(\phi_1, \phi_2)(x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

$$= \phi_1 \phi_2' - \phi_2 \phi_1'$$

$$= x \left(\cancel{\log x} \cdot \frac{1}{x} + \log x \cdot 1 \right) - x \log x (1)$$

$$= x(1 + \log x) - x \log x$$

$$= x + x \log x - x \log x$$

$$= x$$

$$u_1(x) = \int \frac{-b(x)\phi_2(x)}{W(\phi_1, \phi_2)(x)} dx$$

$$= \int \frac{-x^2 (x \log x)}{x} dx$$

$$= \int - \frac{x^2 \log x}{x} dx$$

$$u_1(x) = - \cancel{\int \log x \cdot x^2 dx} - \int x \log x \cdot x^{-1} dx$$

$$= -\left[\log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3} dx \right] = -\int \log x \cdot 1 \, dx$$

$$= -\left[\log x \cdot \frac{x^3}{3} - \int \frac{x^2}{3} dx \right] = -\left[\log x \cdot x - \int \frac{1}{x} \cdot x \, dx \right]$$

$$= -\log x \cdot \frac{x^3}{3} + \frac{x^3}{9} \quad = -[\log x \cdot x - x]$$

$$= -\log x \cdot x + x$$

$$= \frac{x^3}{3}\left(\frac{1}{3} - \log x \right) \quad = x - x \log x$$

$$u_2(x) = \int \frac{b(x)\phi_1(x)}{W(\phi_1, \phi_2)(x)} dx$$

$$= \int \frac{x^2 \cdot x}{x} dx$$

$$= \int x^2 dx$$

$$= \frac{x^3}{3} x$$

$$\psi_p = u_1\phi_1 + u_2\phi_2$$

$$= (x - x \log x) x + (x) x \log x$$

$$= x^2 - x^2 \log x + x^2 \log x$$

$$= x^2$$

$$\psi(x) = \psi_p + c_1\phi_1 + c_2\phi_2$$

$$\psi(x) = x^2 + c_1 x + c_2 x \log x$$

$$\psi(1) = 1 + c_1 + c_2 \log(1)$$

$$1 = 1 + c_1$$

$$\boxed{c_1 = 0}$$

$\psi'(x) = 2x + c_1 + c_2(1+\log x)$

$\psi'(1) = 2 + c_1 + c_2(1+\log 1)$

$0 = 2 + c_1 + c_2$

$c_1 + c_2 = -2$

$\boxed{c_2 = -2}$

$\psi = x^2 - 2x\log x$

11) Show that there is a basis ϕ_1, ϕ_2 for the sol^n of $x^2y'' + 4xy' + (2+x^2)y = 0$, $x > 0$ of the form $\phi_1(x) = \frac{\psi_1(x)}{x^2}$, $\phi_2(x) = \frac{\psi_2(x)}{x^2}$

Hence, find all the solution of $x^2y'' + 4xy' + (2+x^2)y = x^2$, $x > 0$

Solution :

$L(y) = x^2y'' + 4xy' + (2+x^2)y = 0$ —— ①

Let ϕ be the solution of eq^n $L(y) = 0$

$x^2\phi'' + 4x\phi' + (2+x^2)\phi = 0$ —— ②

Let $v = v(x)$ be a function such that

$v = x^2\phi$

$\Rightarrow \phi = \frac{v}{x^2} = \frac{v(x)}{x^2}$

where the function v is to be determine

since ϕ is sol^n of eq^n ①

we have $\phi = \frac{v(x)}{x^2} = \frac{v}{x^2} = v \cdot x^{-2}$

Now, $\phi' = v \cdot -2 \cdot x^{-3} + v'x^{-2}$

$= -2v\,x^{-3} + v'x^{-2}$

$= v'x^{-2} - 2x^{-3}v$

$\phi'' = v'' \cdot x^{-2} - 2x^{-3}v' - 2x^{-3}v' + 6x^{-4}v$

$= \cancel{v''x^{-2} - 2x^{-3}v' - 2x^{-3}v' - 6x^{-4}v}$

$= v''x^{-2} + 6x^{-4}v - 4x^{-3}v'$

$x^2(v''x^{-2} - 4x^{-3}v' + 6x^{-4}v) + 4x(v'x^{-2} - 2x^{-3}v) + (2+x^2)(vx^{-2}) = 0$

$v'' - 4x^{-1}v' + 6x^{-2}v + 4x^{-1}v' - 8x^{-2}v + 2vx^{-2} + v = 0$

$v'' + v = 0$ —— ③

$r^2 + 1 = 0$

$r^2 = -1 \Rightarrow r = \pm i$

$v_1(x) = \cos x$, $v_2 = \sin x$

$\phi_1(x) = \frac{v_1(x)}{x^2} = \frac{\cos x}{x^2}$

$\phi_2(x) = \frac{v_2(x)}{x^2} = \frac{\sin x}{x^2}$

$\therefore$ ϕ_1, ϕ_2 are solution of $L(y) = 0$.

$\therefore$ ϕ_1, ϕ_2 are L.I.

$\therefore$ $\phi_1(x) = \frac{\cos x}{x^2}$

$\phi_2(x) = \frac{\sin x}{x^2}$

form the basis for the eq^n ①

Now, find the solution of eq^n

$x^2 y'' + 4xy' + (2+x^2)y = x^2$

~~$W(\phi_1, \phi_2)(x) = \Big| \frac{\cos x}{x^2} \quad \frac{\sin x}{x^2}$~~ $\quad \frac{\sin x}{x^2}$

$$W(\phi_1,\phi_2)(x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

$= \phi_1 \phi_2' - \phi_2 \phi_1'$

$= \frac{\cos x}{x^2}\left[\sin x \cdot \cancel{x}\, {-2x^{-3}} + x^2 \cos x\right]$

$- \frac{\sin x}{x^2}\left[\cos x \cdot -2x^{-3} + x^2(\sin x)\right]$

$= \frac{\cos x}{x^2}\left[-\frac{2\sin x}{x^3} \cdot \cancel{x} + \frac{\cos x}{x^2}\right]$

$- \frac{\sin x}{x^2}\left[\frac{2\cos x}{x^3} - \frac{\sin x}{x^2}\right]$

$= \frac{-2\cos x \sin x}{x^5} + \frac{\cos^2 x}{x^4} + \frac{2\sin x \cos x}{x^5} + \frac{\sin^2 x}{x^4}$

$\int uv\,dx = u\int v\,dx - \int(u'\int v\,dx)\,dx$

$= \frac{\cos^2 x}{x^4} + \frac{\sin^2 x}{x^4}$

$= \frac{1}{x^4}(\cos^2 x + \sin^2 x)$

$= \frac{1}{x^4}$

$W(\phi_1,\phi_2)(x) = x^{-4}$

$u_1 = \int \frac{-b(x)\phi_2(x)}{W(\phi_1,\phi_2)}\,dx$

$= \int \frac{-\cancel{x^2} \cdot \frac{\sin x}{x^2}}{x^{-4}}\,dx$

$= \int \frac{-\sin x}{x^{-4}}\,dx$

$= -\int \sin x \cdot x^{4}\,dx$

$= -\left[\sin x \cdot \frac{x^5}{5} - \int \cos x \cdot x^5\right.$

$= -\left[x^4(-\cos x) - \int 4x^3 \cdot (-\cos x)\,dx\right]$

$= -(x^4 \cos x + 4 \cdot \frac{x^4}{4} \cos x)$.

$= x^4 \cos x + x^4 \cos x$

$u_1 = x^2(-\cos x) - \int 2x(-\cos x)\,dx$

$= -x^2\cos x + (+2x\sin x - \int 2\sin x\,dx)$

$= -x^2\cos x + 2x\sin x - \left[2(-\cos x) - \int 0(\cos)\,dx\right]$

$= -x^2\cos x + 2x\sin x + 2\cos x$

$= x^2\cos x - 2x\sin x - 2\cos x$

$u_2 = \int 1 \cdot \frac{\cos x}{x^2} \cdot x^4 \, dx$

$= \int \cos x \cdot x^2 dx$

$= x^2 \sin x - \int 2x \sin x \, dx$

$= x^2 \sin x - (2x(-\cos x) - \int 2(-\cos x) dx)$

$= x^2 \sin x + 2x\cos x - 2\sin x - \int 0 \cdot \sin x \, dx$

$= x^2 \sin x + 2x\cos x - 2\sin x$

$\psi_p = u_1\phi_1 + u_2\phi_2$

$= (x^2\cos x + 2x\sin x - 2\cos x) \frac{\cos x}{x^2}$

$\quad + (x^2 \sin x + 2x\cos x + 2\sin x) \frac{\sin x}{x^2}$

$= \cos^2 x - \frac{2\sin x \cdot \cos x}{x} - \frac{2\cos^2 x}{x^2}$

$\quad + \sin^2 x + \frac{2\cos x \cdot \sin x}{x} + \frac{2\sin^2 x}{x^2}$

$= \cos^2 x + \sin^2 x - \frac{2\cos^2 x}{x^2} + \frac{2\sin^2 x}{x^2}$

$= 1 - \frac{2}{x^2}$

(2) Consider the eq^n $y'' + y = b(x)$ where, $b(x)$ is continuous on $1 \le x < \infty$ satisfying $\int_1^\infty |b(t)| dt < \infty$. Show that a particular sol^n ψ_p is given by

$\psi_p(x) = \int_1^x \sin(x-t)\, b(t)\, dt.$

Let $L(y) = y'' + y = b(x)$ —— ①

Be two linearly independant sol^n of $L(y) = 0$ are

$\phi_1(x) = \cos x, \quad \phi_2(x) = \sin x$

Hence, the particular sol^n ψ_p of $L(y) = b(x)$ is given by

$\psi_p(x) = u_1\phi_1 + u_2\phi_2$

where, $u_1(x) = \int \frac{-b(t)\,\phi_2(t)}{W(\phi_1,\phi_2)(t)} dt$

$u_2(x) = \int \frac{b(t)\,\phi_1(t)}{W(\phi_1,\phi_2)(t)} dt$,

$W(\phi_1,\phi_2)(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x$

$= 1$

$u_1(x) = \int_1^x \frac{-b(s) \sin s}{1} ds$

$u_2(x) = \int_1^x b(s) \cos s \, ds$

Now, find ψ_p of x

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a) Let $L(y) = y''' + a_1(x) y'' + a_2(x) y' + a_3(x) y = 0$ —— ①

Let $\phi = u\phi_1$ be a solution of eqⁿ ①.

$L(\phi) = 0$

$L(u\phi_1) = 0$

$(u\phi_1)''' + a_1(x)(u\phi_1)'' + a_2(x)(u\phi_1)' + a_3(x)u\phi_1 = 0$

$(u\phi_1)' = u\phi_1' + u'\phi_1$

$(u\phi_1)'' = u''\phi_1 + u'\phi_1' + u'\phi_1' + u\phi_1''$

$= u''\phi_1 + 2u'\phi_1' + u\phi_1''$

$(u\phi_1)''' = u'''\phi_1 + u''\phi_1' + 2(u''\phi_1' + u'\phi_1'') + u'\phi_1'' + u\phi_1'''$

$= u'''\phi_1 + 3u''\phi_1' + 3u'\phi_1'' + u\phi_1'''$

$(u'''\phi_1 + 3u''\phi_1' + 3u'\phi_1'' + u\phi_1''') + a_1(x)(u''\phi_1 + 2u'\phi_1' + u\phi_1'') + a_2(x)(u'\phi_1 + u\phi_1') + a_3(x)u\phi_1 = 0$

$u'''\phi_1 + u''(3\phi_1' + a_1(x)\phi_1) + u'(3\phi_1'' + 2a_1(x)\phi_1' + a_2(x)\phi_1) + u(\phi_1''' + a_1(x)\phi_1'' + a_2(x)\phi_1' + a_3(x)\phi_1) = 0.$

$u'''\phi_1 + u''(3\phi_1' + a_1(x)\phi_1) + u'(3\phi_1'' + a_1(x)2\phi_1' + a_2(x)\phi_1) = 0.$

$u' = v$

$u'' = v'$

$u''' = v''$

$v''\phi_1 + v'(3\phi_1' + a_1(x)\phi_1) + v(3\phi_1'' + 2a_1(x)\phi_1' + a_2(x)\phi_1) = 0$ —— ②

which is differentiable eqⁿ of order 2.

Now, we claim that $\left(\frac{\phi_2}{\phi_1}\right)'$ satisfy eqⁿ ②

does we simply prove that,

$v = \left(\frac{\phi_2}{\phi_1}\right)'$

we write $\phi_2 = \frac{\phi_2}{\phi_1} \cdot \phi_1$

$\phi_2 = u\phi_1$

for $u = \frac{\phi_2}{\phi_1}$

$\phi_2 = u\phi_1$ is another solⁿ of $L(y) = 0$

$\therefore u = \frac{\phi_2}{\phi_1} \Rightarrow u' = \left(\frac{\phi_2}{\phi_1}\right)'$

Since, v is a solution of eqⁿ ③.

$\Rightarrow v = \left(\frac{\phi_2}{\phi_1}\right)'$ is a solution of eqⁿ ②.

b) Consider the eqⁿ ②

$v''\phi_1 + v'(3\phi_1' + a_1(x)\phi_1) + v(3\phi_1'' + 2a_1(x)\phi_1' + a_2(x)\phi_1) = 0$

Dividing throughout by ϕ_1

$v'' + \frac{v'}{\phi_1}(3\phi_1' + a_1(x)\phi_1) + \frac{v}{\phi_1}(3\phi_1'' + 2a_1(x)\phi_1' + a_2(x)\phi_1) = 0$

$v'' + A_1(x)v' + A_2(x)v = 0 \quad \text{——} ③$

where, $A_1(x) = \frac{1}{\phi_1}(3\phi_1' + a_1(x)\phi_1)$

$A_2(x) = \frac{1}{\phi_1}(3\phi_1'' + 2a_1(x)\phi_1' + a_2(x)\phi_1)$

Let $\phi = \left(\frac{\phi_2}{\phi_1}\right)'$ we satisfy the eqⁿ ③. hence,

we assume ϕ is known solⁿ of eqⁿ ③.

$\phi'' + A_1(x)\phi' + A_2(x)\phi = 0 \quad \text{——} ④$

Hence, we known another solⁿ of eqⁿ ③

is given by $v = W\phi$.

~~here~~ hence, eqⁿ ③ reduces to

$(W\phi)'' + A_1(x)(W\phi)' + A_2(x)(W\phi) = 0$

$(W\phi)' = W'\phi + W\phi'$

$(W\phi)'' = W''\phi + W'\phi' + W'\phi' + W\phi''$

$= W''\phi + 2W'\phi' + W\phi''$

$(W''\phi + 2W'\phi' + W\phi'') + A_1(x)(W'\phi + W\phi') + A_2(x)(W\phi) = 0$

$W''\phi + W'(2\phi' + A_1(x)\phi) + W(\phi'' + A_1(x)\phi' + A_2(x)\phi) = 0$

(by ④)

$W''\phi + W'(2\phi' + A_1(x)\phi) = 0$.

$W' = t$

$t'\phi + t(2\phi' + A_1(x)\phi) = 0$

which is first order differential eqⁿ.

IMP 14] Two solution of $x^3y''' - 3xy' + 3y = 0,\ x > 0$ are $\phi_1(x) = x,\ \phi_2(x) = x^3$, find the third independent solution.

$x^3y''' - 3xy' + 3y = 0$

$y''' - \frac{3}{x^2}y' + \frac{3}{x^3}y = 0$

$a_1(x) = 0,\quad a_2(x) = \frac{-3}{x^2},\quad a_3(x) = \frac{3}{x^3}$

& given that $\phi_1(x) = x,\ \phi_2(x) = x^3$ are solution of $L(y) = 0$

To find the third solⁿ of $L(y) = 0$

where u is to be determine.

Let $\phi_3 = \phi = u\phi_1$ be the third solⁿ of $L(y) = 0$

where, u is to be determine

$x^3(u\phi_1)''' - 3x(u\phi_1)' + 3(u\phi_1) = 0$.

$(u\phi_1)' = u'\phi_1 + u\phi_1'$

$(u\phi_1)'' = u''\phi_1 + u'\phi_1' + u'\phi_1' + u\phi_1''$

$= u''\phi_1 + 2u'\phi_1' + u\phi_1''$

$(u\phi_1)''' = u'''\phi_1 + u''\phi_1' + 2u''\phi_1' + 2u'\phi_1'' + u'\phi_1'' + u\phi_1'''$

$= u'''\phi_1 + 3u''\phi_1' + 3u'\phi_1'' + u\phi_1'''$

$$x^3(u'''\phi_1 + 3u''\phi_1' + 3u'\phi_1'' + u\phi_1''') - 3x(u'\phi_1 + u\phi_1')$$
$$+ 3u\phi_1 = 0$$

$$u'''x^3\phi_1 + u''(3\phi_1' x^3) + u'(3\phi_1''x^3 - 3x\phi_1)$$
$$+ u(\phi_1'''x^3 - 3x\phi_1' + 3\phi_1) = 0$$

$$u'''\phi_1 + u''(3\phi_1') + u'\left(3\phi_1'' - \frac{3}{x^2}\phi_1\right) = 0$$

Define the new fu^n

$$u' = v$$
$$u'' = v'$$
$$u''' = v''$$

$$v''\phi_1 + v'(3\phi_1') + v\left(3\phi_1'' - \frac{3}{x^2}\phi_1\right) = 0$$

since $\phi_1(x) = x$

$$\Rightarrow \phi_1'(x) = 1$$
$$\Rightarrow \phi_1''(x) = 0$$

$$v''x + v'3(1) + v\left(3(0) - \frac{3}{x^2}x\right) = 0$$

$$xv'' + 3v' - \frac{3}{x}v = 0$$

$$v'' + \frac{3}{x}v' - \frac{3}{x^2}v = 0 \quad \text{——} ②$$

This is the second order diff. eq^n

we know $\phi = \left(\frac{\phi_2}{\phi_1}\right)'$ is a solution of eq^{n}②

we consider ϕ is a known sol^n of eq^{n}②

& assume $v = W\phi$ another sol^n of eq^{n}②

where W is to be determine.

Here, $\phi = \left(\frac{\phi_2}{\phi_1}\right)' = \left(\frac{x^3}{x}\right)' = (x^2)' = 2x$ —— *

$$L(W\phi) = 0$$

$$(W\phi)'' + \frac{3}{x}(W\phi) - \frac{3}{x^2}(W\phi) = 0$$

$$(W\phi)' = W'\phi + W\phi'$$
$$(W\phi)'' = W''\phi_1 + W'\phi' + W'\phi' + W\phi''$$
$$= W''\phi_1 + 2W'\phi' + W\phi''$$

$$(W''\phi + 2W'\phi' + W\phi'') + \frac{3}{x}(W'\phi + W\phi') - \frac{3}{x^2}(W\phi) = 0$$

$$W''\phi_1 + W'\left(2\phi' + \frac{3}{x}\phi\right) + W\left(\phi'' + \frac{3}{x}\phi' - \frac{3}{x^2}\phi\right) = 0$$

$$W''\phi + W'\left(2\phi' + \frac{3}{x}\phi\right) = 0$$

$$\phi = 2x \quad \text{— (by *)}$$
$$\phi' = 2$$

$$W''(2x) + W'\left(2(2) + \frac{3}{x}(2x)\right) = 0$$

$$2xW'' + W'(4+6) = 0$$
$$2xW'' + 10W' = 0$$
$$xW'' + 5W' = 0$$

put $W' = t$

$$xt' + 5t = 0$$
$$t' + \frac{5}{x}t = 0$$

This is the first order diff. eq^n.

$$\frac{dt}{dx} + \frac{5}{x}t = 0$$

$$\frac{dt}{dx} = -\frac{5}{x}t$$

$$\frac{dt}{t} = \frac{-5}{x}dx$$

$$\Rightarrow \frac{-1}{5t}dt = \frac{1}{x}dx$$

Integrating

$-\frac{1}{5}\log t = \log x + \log c.$

$\log t = -5\log x + \log c.$

$\log t = \log(x^{-5}c)$

$t = x^{-5}c$

$t = \frac{c}{x^5}$

$W' = t$

Integrating t

$W = \int t$

$= \int \frac{c}{x^5}\,dx$

$= \int c\,x^{5}\,dx$

$= c.\frac{x^{-4}}{-4}$

$= \frac{-c}{4x^4}$

$W = \frac{c_1}{x^4}$, where $c_1 = \frac{-c}{4}$

but $V = W\phi$

$= \frac{c_1}{x^4}.2x$

$= \frac{2c_1}{x^3}$

$V = \frac{c_2}{x^3}$ where, $c_2 = 2c_1$

$u' = V$

Integrating

$u = \int V$

$u = \int \frac{c_2}{x^3}\,dx$

$= \int c_2.x^{-3}\,dx$

$= c_2.\frac{x^{-2}}{-2}$

$= \frac{c_2}{-2x^2}$

$u = \frac{c_3}{x^2}$ $\qquad c_3 = \frac{-c_2}{2}$

$u = \frac{1}{x^2}$

This implies the consequently. the solⁿ of eqⁿ ① is given by

$\phi_3 = \phi = u\phi,$

$= \frac{1}{x^2}.x$

$\phi_3 = \frac{1}{x}$

which is the third independant solution of eqⁿ ①.

Consider the eqⁿ $y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0,\ x>0$

1) Show that there is a solⁿ of the form x^r where, r is const.

2) Find two L.I. solⁿ for $x>0$ & proof that they are L.I.

3) Find the two solⁿ ϕ_1 & ϕ_2 satisfying $\phi_1(1)=1$, $\phi_2(1)=0$, $\phi_1'(1)=0$, $\phi_2'(1)=1$

$x^2y''+xy'-y=0$

Given

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$$

a) Let $\phi(x) = x^r$ be a solution of $L(y) = 0$.

$\phi_1(x) = x$ & $\phi_2(x) = \frac{1}{x}$ are two sol^n of $L(y)=0$

$\phi(x) = c_1\phi_1 + c_2\phi_2$

$\phi(x) = c_1 x + \frac{c_2}{x}$

$\phi'(x) = c_1 \cancel{\frac{x}{x}} - c_2\frac{1}{x^2}$

$\phi''(x) + \frac{1}{x}\phi'(x) - \frac{1}{x^2}\phi(x) = 0$

$(x^r)'' + \frac{1}{x}(x^r)' - \frac{1}{x^2}(x^r) = 0$

$(x^r)' = r \cdot x^{r-1}$, $(x^r)'' = r(r-1)x^{r-2}$

$r \cdot \cancel{x}(r-1)x^{r-2} + r \cdot x^{r-2} - x^{r-2} = 0$

$x^{r-2}(r(r-1) + r - 1) = 0$

$x^{r-2} \neq 0$, $r(r-1) + 1(r-1) = 0$

$(r+1)(r-1) = 0$

$r = 1, -1$

$\phi_1 = e^x$; $\phi_2 = e^{-x}$

In $x^r \Rightarrow \phi_1 = x$, $\phi_2 = \frac{1}{x}$

$\phi = c_1\phi_1 + c_2\phi_2$

$= c_1 x + c_2 \frac{1}{x}$ —— ①

b) Diff. eq^n ① w.r.t. x

$c_1 - c_2\frac{1}{x^2} = 0$

again, diff. w.r.t. x

$-2c_2\frac{1}{x^3} = 0$

$c_2 = 0$

$\Rightarrow c_1 = 0$.

$\therefore \phi$ is a L.I. solution

c) $\phi_1 = c_1 x + c_2\frac{1}{x}$

$\phi_1' = c_1 - c_2 \frac{1}{x^2}$

$\phi_1(1) = c_1 + c_2$

$c_1 + c_2 = 1$ —— ②

$\phi_1'(1) = c_1 - c_2$

$c_1 - c_2 = 0$ —— ③

eq^n ② + eq^n ③.

$2c_1 = 1$

$c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2}$.

$\phi_2 = c_1 x + c_2 \frac{1}{x}$

$\phi_2' = c_1 - c_2 \frac{1}{x^2}$

$\phi_2(1) = c_1 + c_2$

$c_1 + c_2 = 0$ —— ④

$\phi_2'(1) = c_1 - c_2$

$c_1 - c_2 = 1$ —— ⑤

eq^n ④ + ⑤

$2c_1 = +1$

$c_1 = +\frac{1}{2}$

$c_2 = -\frac{1}{2}$

$\phi_1 = \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{x^2}$

$\phi_2 = \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{x^2}$

* Homogeneous eq^n with analytic coefficient &

Analytic function : Continuously differentiable.

Def^n :

A function $f(x)$ is said to be analytic in a neighbourе & point $x = x_0$ if the derivative & all order exist in the neighbourhood & x_0 if $f(x)$ & passes convergent power series expansion about x_0 then

i.e. $f(x)$ is the analytic function at x_0 then it can be represented in the form

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

where, c_k are constant & the series converges $|x - x_0| < r_0$, $x_0 > 0$

- Def^n :

A series $\sum_{k=0}^{n} c_k (x - x_0)^k$ is said to be convergent if $\lim_{n \to \infty} \sum_{k=0}^{n} c_k (x - x_0)^k$ exist

- Ratio Test :-

Consider a series $\sum_{k=0}^{\infty} |c_k|$

where, c_k are complex

If $|c_k| > 0$ & $\left| \frac{c_{k+1}}{c_k} \right| \to L$ as $k \to \infty$

then the series convergent. If $L < 1$ & divergent if $L > 1$.

Analytic function → locally given by convergent power series

Real analytic function | Complex f

Both are infinitely differentiable

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IMP

1] Find two ~~p~~ L.I. power series of eq^n $y''+y=0$ near ordinary pt $x=0$.

Solution:

we see that the eq^n $y''+y=0$ has no singularity in the finite plane.

Since, the coefficient $a_1(x)=0$, $a_2(x)=1$ are analytic for all x_0.

It ~~terms~~ that the solution are also analytic at the ordinary point $x_0=0$.

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k \quad \text{---} *$$

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x)^k \quad (\because x_0=0) \quad \text{---} ②$$

be a power series solution of eq^n ①

i.e. $\phi''(x)+\phi(x)=0$ *

we have, $\phi(x) = \sum_{k=0}^{\infty} c_k x^k$

$$\phi(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

$$\phi'(x) = \sum_{k=1}^{\infty} c_k k x^{k-1}$$

$$= \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k \quad (\text{replace } k \text{ by } k+1)$$

$c_1 + 2c_2 x + \dots$

$$\phi''(x) = \sum_{k=2}^{\infty} c_k \cdot k(k-1) x^{k-2}$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k \quad (\text{replace } k \text{ by } k+2)$$

$2c_2 + 3(2)c_3 x$

substitute this value in eq^n * weget.

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k = 0 + 0x + 0x^2 + \dots$$

$$\sum_{k=0}^{\infty} [c_{k+2}(k+2)(k+1) + c_k] x^k = 0 + 0x + 0x^2 + \dots + 0x^k$$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1) + c_k = 0$$

i.e. $c_{k+2}(k+2)(k+1) + c_k = 0$, $k=0,1,2,\dots$

$$c_{k+2} = \frac{-c_k}{(k+2)(k+1)} \quad , k = 0, 1, 2, \dots$$

These eqⁿ is called recurrence relation.

by given $k = 0, 1, \dots$

we obtain the constant.

For $k = 0$ then

$$c_2 = \frac{-c_0}{(2)(1)} = \frac{-c_0}{2} \quad \text{— ①}$$

For $k = 1$

$$c_3 = \frac{-c_1}{(3)(2)} \quad \text{— ②}$$

For $k = 2$

$$c_4 = \frac{-c_2}{(4)(3)} = \frac{-(-c_0)}{2\cdot4\cdot3} = \frac{c_0}{2\cdot3\cdot4} \quad \text{— ③ by ①}$$

For $k = 3$

$$c_5 = \frac{-c_3}{(5)(4)} = \frac{-(-c_1)}{2\cdot3\cdot4\cdot5} = \frac{c_1}{2\cdot3\cdot4\cdot5} \quad \text{— by ② — iv}$$

For $k = 4$

$$c_6 = \frac{-c_4}{(6)(5)} = \frac{-c_0}{2\cdot3\cdot4\cdot5\cdot6} \quad \text{by ③}$$

For $k = 5$

$$c_7 = \frac{-c_5}{(7)(6)} = \frac{-c_1}{2\cdot3\cdot4\cdot5\cdot6\cdot7} \quad \text{by iv}$$

$c_2 = \frac{-c_0}{2!}$ $\quad$ $c_3 = \frac{-c_1}{3!}$

$c_4 = \frac{c_0}{4!}$ $\quad$ $c_5 = \frac{c_1}{5!}$

$c_6 = \frac{-c_0}{6!}$ $\quad$ $c_7 = \frac{-c_1}{7!}$

& so on

In general we obtain

$$c_{2k} = \frac{(-1)^k c_0}{(2k)!}$$

$$c_{2k+1} = \frac{(-1)^k c_1}{(2k+1)!}$$

From eqⁿ ②, we get

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$$

$$\phi(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + \dots$$

$$= c_0 + (c_2x^2 + c_4x^4 + c_6x^6 + \dots) + (c_1x + c_3x^3 + c_5x^5 + c_7x^7 + \dots)$$

$$= c_0 + \sum_{k=1}^{\infty} c_{2k}x^{2k} + c_1x + \sum_{k=1}^{\infty} c_{2k+1}x^{2k+1}$$

$$= c_0 + \sum_{k=1}^{\infty} \frac{(-1)^k c_0}{(2k)!} x^{2k} + c_1x + \sum_{k=1}^{\infty} \frac{(-1)^k c_1}{(2k+1)!} x^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k c_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k c_1}{(2k+1)!} x^{2k+1}$$

$$\therefore \phi(x) = c_0\phi_1(x) + c_1\phi_2(x)$$

where, $\phi_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$ — ③

$\phi_2(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ — ④

From eqⁿ ③, suppose

$$\phi_1(x) = \sum_{k=0}^{\infty} d_k$$

where, $d_k = \frac{(-1)^k \cdot x^{2k}}{(2k)!}$

by the ratio test.

$$\lim_{k\to\infty}\left|\frac{d_{k+1}}{d_k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{\frac{(-1)^{k+1}\, x^{2(k+1)}}{2(k+1)!}}{\frac{(-1)^k x^{2k}}{(2k)!}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(-1)^k(-1)\, x^{2k}\cdot x^2}{\underset{(2k+2)(2k+1)2k!}{2(k+1)!}}\cdot\frac{(2k)!}{(-1)^k x^{2k}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{(-1)' x^2}{(2k+2)(2k+1)}\right|$$

$$\therefore \lim_{k\to\infty}\left|\frac{d_{k+1}}{d_k}\right| = 0 \leq 1 \quad \text{(convergent)}$$

This implies $\phi_1(x)$ has convergent power series expansion.
Similarly we can show that the function $\phi_2(x)$ has convergent power series expansion for $|x| < \infty$.

Find two linearly independant power series soln in ~~the~~ power of x of the following eqn.

$y'' - x^2 y = 0$ at ordinary pt $x = 0$.

we see that the eqn $y'' - x^2 y = 0$ — ①
has

here $a_1(x) = 0$ & $a_2(x) = -x^2$
are all analytic for x_0.
The solution are also analytic at the ordinary point $x_0 = 0$.

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k \quad \text{— ②}$$

be a power series soln of eqn ①.
i.e. $\phi''(x) - x^2\phi(x) = 0$ — $*$
we have

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots$$

$$\phi'(x) = \sum_{k=1}^{\infty} c_k\, k\, x^{k-1}$$

$$= \sum_{k=0}^{\infty} c_{k+1}(k+1)\, x^{k}$$

$$\phi''(x) = \sum_{k=2}^{\infty} c_k \cdot k(k-1)\, x^{k-2}$$

$$= \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)\, x^k$$

substituting this values in eqn $*$

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - x^2\left(\sum_{k=0}^{\infty} c_k x^k\right) = 0 + 0x + 0x^2 + \ldots$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} C_k x^{k+2} = 0 + 0x + \dots 0x^k$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=2}^{\infty} C_{k-2}\, x^k = 0$$

(replace k by $k-2$)

$$\underset{k=0}{2C_2} + \underset{k=1}{6C_3 x} + \sum_{k=2}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=2}^{\infty} C_{k-2}\, x^k = 0$$

$$2C_2 + 6C_3 x + \sum_{k=2}^{\infty}\left(C_{k+2}(k+2)(k+1) - C_{k-2}\right)x^k = 0$$

$0 + 0x + \dots + 0x^k$

$\Rightarrow$ $2C_2 = 0$, $6C_3 x = 0$

$\boxed{C_2 = 0}$ $\boxed{C_3 = 0}$

$$C_{k+2}(k+2)(k+1) - C_{k-2} = 0$$

$$C_{k+2} = \frac{C_{k-2}}{(k+2)(k+1)}, \quad k = 2, 3, \dots$$

This eq^n is called reccurrence relation

by given $k = 2, 3, \dots$

we obtain the constant

For $k=2$

$C_4 = \dfrac{C_0}{(4)(3)} = \dfrac{C_0}{12}$

For $k=3$

$C_5 = \dfrac{C_1}{(5)(4)} = \dfrac{C_1}{20}$

For $k=4$

$C_6 = \dfrac{C_2}{(6)(5)} = 0$

For $k=5$

$C_7 = \dfrac{C_3}{(7)(6)} = 0$

For $k=6$

$C_8 = \dfrac{C_4}{(8)(7)} = \dfrac{C_0}{4 \cdot 3 \cdot 8 \cdot 7}$

For $k=7$

$C_9 = \dfrac{C_5}{(9)(8)} = \dfrac{C_1}{5 \cdot 4 \cdot 8 \cdot 9}$

For $k=8$

$C_{10} = \dfrac{C_6}{(10)(9)} = 0$

For $k=9$

$C_{11} = \dfrac{C_7}{(11)(10)} = 0$

For $k=10$

$C_{12} = \dfrac{C_8}{(12)(11)} = \dfrac{C_0}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12}$

For $k=11$

$C_{13} = \dfrac{C_9}{(13)(12)} = \dfrac{C_1}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13}$

~~$C_{4k+2} = 0$~~ , ~~$C_{4k+3} = 0$~~

$C_2 = C_3 = C_6 = C_7 = C_{10} = C_{11} = \dots$

$C_{4k-1} = 0$

$C_{4k-2} = 0$, $k = 1, 2, \dots$

$C_4 = \dfrac{C_0}{4 \cdot 3}$, $C_8 = \dfrac{C_0}{3 \cdot 4 \cdot 7 \cdot 8}$, $C_{12} = \dfrac{C_0}{3 \cdot 4 \cdot 7 \cdot 8 \cdot 11 \cdot 12}$, $\dots$

$$C_{4k} = \frac{C_0}{(3)(4)(7)(8)(11)(12)\dots(4k-1)(4k)}, \quad k = 1, 2, \dots$$

$C_5 = \dfrac{C_1}{5 \cdot 4}$, $C_9 = \dfrac{C_1}{4 \cdot 5 \cdot 8 \cdot 9}$, $C_{13} = \dfrac{C_1}{4 \cdot 5 \cdot 8 \cdot 9 \cdot 12 \cdot 13}$, $\dots$

$$C_{4k+1} = \frac{C_1}{(4)(5)(8)(9)\dots(4k)(4k+1)}$$

From eq^n ②

$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$

$\phi(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + c_{12} x^{12} + \dots$

$\phi(x) = c_0 + c_1 x + c_4 x^4 + c_5 x^5 + c_8 x^8 + c_9 x^9 + c_{12} x^{12} + \dots$

$(c_2 = c_3 = c_6 = c_7 = c_{10} = c_{11} = c_{14} = 0)$

$= c_0 + \sum_{k=1}^{\infty} c_{4k} x^{4k} + c_1 x + \sum_{k=1}^{\infty} c_{4k+1} x^{4k+1}$

$= c_0 + \sum_{k=1}^{\infty} \frac{c_0}{(3)(4)(7)(8)(11)(12)\dots(4k-1)(4k)} x^{4k} +$

$c_1 x + \sum_{k=1}^{\infty} \frac{c_1}{(4)(5)(8)(9)(12)(13)\dots(4k)(4k+1)} x^{4k+1}$

$\phi(x) = c_0\left[1 + \sum_{k=1}^{\infty} \frac{x^{4k}}{(3)(4)(7)(8)(11)(12)\dots(4k-1)(4k)}\right] +$

$c_1\left[x + \sum_{k=1}^{\infty} \frac{x^{4k+1}}{(4)(5)(8)(9)(12)(13)\dots(4k)(4k+1)}\right]$

$\therefore \phi(x) = c_0 \phi_1(x) + c_1 \phi_2(x)$

where, $\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{4k}}{(3)(4)(7)(8)\dots(4k-1)(4k)}$

$\phi_2(x) = x + \sum_{k=1}^{\infty} \frac{x^{4k+1}}{(4)(5)(8)(9)\dots(4k)(4k+1)}$

Suppose $\phi_1(x) = 1 + \sum_{k=1}^{\infty} d_k$

where, $d_k = \frac{x^{4k}}{(3)(4)(7)(8)\dots(4k-1)(4k)}$ $4k+4-1$??

$\lim_{k\to\infty} \left|\frac{d_{k+1}}{d_k}\right| = \left|\frac{\frac{x^{4(k+1)}}{(3)(4)(7)(8)\dots(4k+3)(4k+4)}}{\frac{x^{4k}}{(3)(4)(7)(8)\dots(4k-1)(4k)}}\right|$

$\lim_{k\to\infty} \left|\frac{d_{k+1}}{d_k}\right| = \left|\frac{x^{4k+4}}{(3)(4)(7)(8)\dots(4k+3)(4k+4)} \cdot \frac{(3)(4)}{x^{4k}}\right|$

$= \left|\frac{x^{4k+4}}{(3)(4)(7)(8)\dots(4k+3)(4k+4)} \cdot \frac{(3)(4)(7)(8)\dots(4k-1)(4k)}{x^{4k}}\right|$

$(4k-1)(4k)(4k+1)(4k+2)(4k+3)(4k+4)$

$= \left|\frac{x^{4k}}{(4k+1)(4k+2)(4k+3)(4k+4)}\right|$

$= 0$

$\therefore \lim_{k\to\infty} \left|\frac{d_{k+1}}{d_k}\right| = 0 < 1$ convergent.

This implies $\phi_1(x)$ has convergent power series expansion similarly, we can show that the function $\phi_2(x)$ has convergent power series expansion for $|x| < \infty$

IMP
1) $y'' - xy' + y = 0$
2) $y'' + 3x^2 y' - xy = 0$
3) $y'' + x^3 y' + x^2 y = 0$
4) $y'' - xy = 0$

$c_{3k} = \frac{c_0}{2\cdot 3\cdot 5\cdot 6\dots(3k-1)(3k)}$

$c_{3k+1} = \frac{c_1}{3\cdot 4\cdot 6\cdot 7\dots 3k(3k+1)}$

* The Legendre equation :-

One of the second order linear eq^n with analytic coefficient is the legendre eq^n is given by

$$L(y) = (1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0 \quad \text{--- (1)}$$

where, α is constant.

we see that $x = 1$. or $x = -1$ are singular points of legendary eq^n.

we write eq^n (1) as

$$y'' - \frac{2x}{(1-x^2)} y' + \frac{\alpha(\alpha+1)}{(1-x^2)} y = 0$$

i.e. $y'' + a_1(x)y' + a_2(x)y = 0$

where, $a_1(x) = \frac{-2x}{(1-x^2)}$ & $a_2(x) = \frac{\alpha(\alpha+1)}{(1-x^2)}$

we see that,

$\lim_{x\to\infty} (x\pm1)a_1(x)$ & $\lim_{x\to\infty} (x\pm1)^2 a_2(x)$

exists & finite.

we show that. the $x = \pm1$ are regular singular point of the legendre eq^n

This implies $a_1(x)$ & $a_2(x)$ have convergent power series expansion on $|x|<1$

Ex. 1) Show the legendre eq^n written in the form

1) $y'' + a_1(x)y' + a_2(x)y = 0$

where, $a_1(x)$ & $a_2(x)$ have convergent power series expansion.

2) compute two L.I. sol^n for $|x|<1$

3) Show that the α is non-negative integer n there is a polynomial sol^n of degree n.

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$

$(1-x^2)^{-1} = 1 + x^2 + x^4 + x^6 + \dots$

1) Consider the Legendre eq^n

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

where, α is contant.

$$\therefore y'' - \frac{2x}{(1-x^2)} y' + \frac{\alpha(\alpha+1)}{(1-x^2)} y = 0$$

i.e. we have

$$y'' + a_1(x)y' + a_2(x)y = 0$$

where, $a_1(x) = \frac{-2x}{(1-x^2)}$, $a_2(x) = \frac{\alpha(\alpha+1)}{(1-x^2)}$

then the function $a_1(x), a_2(x)$ are analytic at $x=0$

Since, $a_1(x), a_2(x)$ have power series expansion which converges for $|x|<1$.

$$\begin{aligned} a_1(x) &= -2x(1-x^2)^{-1} \\ &= -2x[1 + x^2 + x^4 + x^6 + \dots] \\ &= -2[x + x^3 + x^5 + x^7 + \dots] \end{aligned}$$

$$a_1(x) = -2\sum_{k=0}^{\infty} x^{2k+1}$$

$$\begin{aligned} a_2(x) &= \alpha(\alpha+1)(1-x^2)^{-1} \\ &= \alpha(\alpha+1)[1 + x^2 + x^4 + x^6 + \dots] \\ &= \alpha(\alpha+1)\sum_{k=0}^{\infty} x^{2k} \end{aligned}$$

which is convergent for $|x|<1$

2) Let ϕ be any solution of Legendre eq^n for $|x|<1$ & suppose that

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k \quad (\because x_0 = 0)$$

we must have

$$(1-x^2)\phi'' - 2x\phi' + \alpha(\alpha+1)\phi = 0$$

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$$\phi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$\phi'(x) = \sum_{k=1}^{\infty} C_k k \cdot x^{k-1}$$

$$= \sum_{k=0}^{\infty} C_{k+1} (k+1) x^k$$

$$\phi''(x) = \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2}$$

$$= \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^k$$

$$(1-x^2) \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^k - 2x \sum_{k=0}^{\infty} C_{k+1}(k+1) x^k$$

$$+ \alpha(\alpha+1) \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^k - \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^{k+2}$$

$$-2 \sum_{k=0}^{\infty} C_{k+1}(k+1) x^{k+1} + \alpha(\alpha+1) \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^k - \sum_{k=2}^{\infty} C_k k(k-1) x^k -$$

$$2 \sum_{k=1}^{\infty} C_k (k) x^k + \alpha(\alpha+1) \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\Rightarrow 2C_2 + C_3(3)(2)x + \sum_{k=2}^{\infty} C_{k+2}(k+2)(k+1) x^k -$$

$$- \sum_{k=2}^{\infty} C_k k(k-1) x^k - 2C_1 x - 2 \sum_{k=2}^{\infty} C_k (k) x^k +$$

$$+ \alpha(\alpha+1) C_0 + \alpha(\alpha+1) C_1 x + \alpha(\alpha+1) \sum_{k=2}^{\infty} C_k x^k = 0$$

$$\Rightarrow 2C_2 + \alpha(\alpha+1) C_0 + [C_3(3)(2) - 2C_1 + \alpha(\alpha+1) C_1] x$$

$$+ \sum_{k=2}^{\infty} [C_{k+2}(k+2)(k+1) - C_k k(k-1) - 2C_k \cdot k$$

$$+ \alpha(\alpha+1) C_k] x^k = 0$$

For $k=3$

$$C_5 = \frac{C_3(3-\alpha)(4+\alpha)}{(5)(4)}$$

$$= -\frac{(\alpha-1)(\alpha+2)(3-\alpha)(4+\alpha)}{5\cdot4\cdot3\cdot2}C_1$$

$$= \frac{(\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)}{5\cdot4\cdot3\cdot2}C_1$$

$k=4$

$$C_6 = \frac{C_4(4-\alpha)(5+\alpha)}{(6)(5)}$$

$$= \frac{\alpha(\alpha+1)(\alpha-2)(\alpha+3)(4-\alpha)(5+\alpha)}{6\cdot5\cdot4\cdot3\cdot2}C_0$$

$$= -\frac{\alpha(\alpha+1)(\alpha-2)(\alpha+3)(\alpha-4)(\alpha+5)}{6\cdot5\cdot4\cdot3\cdot2}C_0$$

$k=5$

$$C_7 = \frac{C_5(5-\alpha)(6+\alpha)}{(7)(6)}$$

$$= \frac{(\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)(5-\alpha)(6+\alpha)}{7\cdot6\cdot5\cdot4\cdot3\cdot2}C_1$$

$$= -\frac{(\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)(\alpha-5)(\alpha+6)}{7\cdot6\cdot5\cdot4\cdot3\cdot2}C_1$$

$k=6$

$$C_8 = \frac{C_6(6-\alpha)(7+\alpha)}{(8)(7)}$$

$$= -\frac{\alpha(\alpha+1)(\alpha-2)(\alpha+3)(\alpha-4)(\alpha+5)(6-\alpha)(7+\alpha)}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}C_0$$

$$= \frac{\alpha(\alpha+1)(\alpha-2)(\alpha+3)(\alpha-4)(\alpha+5)(\alpha-6)(\alpha+7)}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}C_0$$

$k=7$

$$C_9 = \frac{C_7(7-\alpha)(8+\alpha)}{(9)(8)}$$

$$C_9 = -\frac{(\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)(\alpha-5)(\alpha+6)(7-\alpha)(8+\alpha)}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}C_1$$

$$= \frac{(\alpha-1)(\alpha+2)(\alpha-3)(\alpha+4)(\alpha-5)(\alpha+6)(\alpha-7)(\alpha+8)}{9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2}C_1$$

$$C_{2k} = \frac{(-1)^k[\alpha+(2k-1)][\alpha+(2k-3)]\ldots(\alpha+1)\alpha(\alpha-2)(\alpha-2k+2)C_0}{(2k)!}$$

$$C_{2k+1} = \frac{(-1)^k[(\alpha+2k)(\alpha+2k-2)\ldots(\alpha+2)(\alpha-1)\ldots(\alpha-2k+1)C_1}{(2k+1)!}$$

$$\phi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$= C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + C_7x^7 + C_8x^8 + C_9x^9 + \ldots$$

$$= C_0 + [C_2x^2 + C_4x^4 + C_6x^6 + C_8x^8 + \ldots] + C_1x + [C_3x^3 + C_5x^5 + C_7x^7 + \ldots]$$

$$= C_0 + \sum_{k=1}^{\infty} C_{2k}x^{2k} + C_1x + \sum_{k=1}^{\infty} C_{2k+1}x^{2k+1}$$

$$= C_0 + \sum_{k=1}^{\infty} \frac{(-1)^k[\alpha+(2k-1)][\alpha+(2k-3)]\ldots(\alpha+1)\alpha(\alpha-2)(\alpha-2k+2)C_0}{(2k)!}x^{2k}$$

$$+ C_1x + \sum_{k=1}^{\infty} \frac{(-1)^k[(\alpha+2k)(\alpha+2k-2)\ldots(\alpha+2)(\alpha-1)\ldots(\alpha-2k+1)C_1}{(2k+1)!}x^{2k+1}$$

$$\phi(x) = C_0\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k[\alpha+(2k-1)][\alpha+(2k-3)]\ldots(\alpha-2k+2)}{(2k)!}x^{2k}\right]$$

$$+ C_1\left[x + \sum_{k=1}^{\infty} \frac{(-1)^k[(\alpha+2k)(\alpha+2k-2)\ldots(\alpha-2k+1)}{(2k+1)!}x^{2k+1}\right]$$

Here, $\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k [\alpha+(2k-1)][\alpha+(2k-3)]\ldots(\alpha-2k+2)}{(2k)!} x^{2k}$

$\phi_2(x) = x + \sum_{k=1}^{\infty} \frac{(-1)^k [(\alpha+2k)(\alpha+2k-2)\ldots(\alpha-2k+1)]}{(2k+1)!} x^{2k+1}$

Now, $\phi_1(0) = 1$, $\phi_2(0) = 0$

$\phi_1'(0) = 0$, $\phi_2'(0) = 1$

$$W(\phi_1,\phi_2)(x) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$= 1$

$\neq 0$

i.e. $\phi_1(x)$ & $\phi_2(x)$ are L.I.

$W(\phi_1,\phi_2)(x) = e^{-a_1(x-x_0)} \; W(x_0)$

$= e^{-a_1(x)} \; (1)$

$= e^{-a_1(x)}$

$\neq 0$

i.e. ϕ_1, ϕ_2 are L.I.

3) 1) If α is non-negative even integer

$\alpha = n = 2m$, $m = 0,1,2,\ldots$

$\phi_1(x) = \cancel{c_0} + c_2x^2 + c_4x^4 + c_6x^6 + \ldots$

$\phi_1(x) = c_0$

then for ,

$\alpha = 0$

$\phi_1(x) = c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \ldots$

$\phi_1(x) = c_0$

$\phi_1(x) = 1 = P_0(x)$

$\boxed{P_0(x) = 1}$

Then, for $\alpha = 2$.

$\phi_1(x) = c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \ldots$

$= c_0 + \left[\frac{-2(3)c_0}{2} \cdot x^2\right] + 0 + 0 + \ldots$

$= c_0 - \frac{6c_0}{2} \cdot x^2$

$\phi_1(x) = c_0(1 - 3x^2)$

$P_2(x) = 1 - 3x^2$

for $\alpha = 4$

$\phi_1(x) = c_0 + c_2x^2 + c_4x^4 + c_6x^6 + \ldots$

$= c_0 + \left[\frac{-4(5)c_0}{2} x^2\right] + \left[\frac{4(5)(2)(7)c_0}{4\cdot 3\cdot 2} x^4\right]$

$+ 0 + 0 + \ldots$

$= c_0 - 10c_0x^2 + \frac{35}{3} c_0x^4$

$= c_0\left(1 - 10x^2 + \frac{35}{3} x^4\right)$

$P_4(x) = 1 - 10x^2 + \frac{35}{3} x^4$ & so on

This implies that ϕ_1 has only finite no. of non zero term moreover $\phi_1(x)$ is a poly of degree n containing only even power of x but the solution ϕ_2 is not a poly. in this case, since none of the coefficient in the series is vanish.

Now, if α is non-negative integer.

$\alpha = n = 2m+1$, $m = 0,1,2,\ldots$

Then, for $\alpha = 1$

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$$\phi_2(x) = c_1 x + c_3 x^3 + c_5 x^5 + \dots$$
$$= c_1 x + \left(\frac{-(3)(0)}{6} c_1\right) x^3 + 0 + \dots$$
$$= c_1 x + 0$$
$$= c_1 x$$
$$P_1(x) = x$$

For $\alpha = 3$

$$\phi_2(x) = c_1 x + c_3 x^3 + c_5 x^5 + \dots$$
$$= c_1 x - \frac{(5)(2)}{6} c_1 x^3 + 0 + 0 + \dots$$
$$= c_1 x - \frac{5}{3} c_1 x^3$$
$$\phi_2(x) = c_1\left(x - \frac{5}{3} x^3\right)$$
$$P_3(x) = x - \frac{5}{3} x^3 \qquad \text{\& so on}$$

For ~~$\alpha = 5$~~

This implies ϕ_2 is a poly. of degree n having only odd power of x & ϕ_1 is not a polynomial in this case, since none of the coefficient in the series ϕ_1 vanishes.

Hence, the proof.

Eg 1) Given differential eq^n is

$$y'' - xy' + y = 0 \quad \text{——} \ (1)$$

Here, $a_1(x) = -x$ & $a_2(x) = 1$ are all analytic for x_0.

The solution are also analytic at the ordinary point $x_0 = 0$.

$$\phi(x) = \sum_{k=0}^{\infty} C_k (x - x_0)^k$$

$$\phi(x) = \sum_{k=0}^{\infty} C_k x^k \quad \text{——} \ (2)$$

be a power series solution of eq^n (1).

i.e. $\phi''(x) - x\phi'(x) + \phi(x) = 0$ —— *

we have

$$\phi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$\phi'(x) = \sum_{k=1}^{\infty} C_k k x^{k-1}$$

$$\phi'(x) = \sum_{k=0}^{\infty} C_{k+1} (k+1) x^k$$

$$\phi''(x) = \sum_{k=2}^{\infty} C_k k(k-1) x^{k-2}$$

$$= \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1) x^k$$

Substituting this values in eq^n *

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - x\left(\sum_{k=0}^{\infty} C_{k+1}(k+1)x^k\right) + \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} C_{k+1}(k+1)x^{k+1} + \sum_{k=0}^{\infty} C_k x^k = 0$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} C_k\, k x^k + \sum_{k=0}^{\infty} C_k x^k = 0$$

$$2C_2 + \sum_{k=1}^{\infty} C_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} C_k\, k x^k + \sum_{k=1}^{\infty} C_k x^k + C_0 = 0$$

$$2C_2 + C_0 + \sum_{k=1}^{\infty} \left(C_{k+2}(k+2)(k+1) - C_k k + C_k\right) x^k = 0$$

$$\cancel{3C_0 + \sum_{k=1}^{\infty} C_k[(k+2)(k+1) - k+1]\, x^k = 0}$$

$$\cancel{3C_0 = 0}$$

$$\cancel{\Rightarrow C_0 = 0}$$

$$\cancel{C_k[(k+2)(k+1) - (k+1)] = 0}$$

$$\cancel{C_k[(k+1)(k+2) - k+}$$

$$2C_2 + C_0 + \left[\sum_{k=1}^{\infty} C_{k+2}(k+2)(k+1) - (k-1)C_k\right] x^k = 0$$

$$2C_2 + C_0 = 0$$

$$\Rightarrow C_2 = \frac{-C_0}{2}$$

$$C_{k+2}(k+2)(k+1) - (k-1)C_k = 0$$

$$C_{k+2}(k+2)(k+1) = (k-1)C_k$$

$$C_{k+2} = \frac{(k-1)C_k}{(k+2)(k+1)}, \quad k = 1, 2.$$

This eq^n is called recurrence relation by given $k = 1, 2, \ldots$

For $k = 1$

$$C_3 = 0$$

For $k = 2$

$$C_4 = \frac{C_2}{(4)(3)} = \frac{-C_0}{4\cdot 3\cdot 2}$$

For $k = 3$

$$C_5 = \frac{2C_3}{(5)(4)} = 2(0) = 0$$

For $k = 4$

$$C_6 = \frac{3C_4}{6\cdot 5} = \frac{-3C_0}{6\cdot 5\cdot 4\cdot 3\cdot 2}$$

For $k=5$

$$c_7 = \frac{4c_5}{(7)(6)} = 0$$

For $k=6$

$$c_8 = \frac{5c_6}{(8)(7)} = \frac{-5\cdot 3\, c_0}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

For $k=7$

$$c_9 = \frac{6c_7}{(9)(8)} = 0$$

For $k=8$

$$c_{10} = \frac{7c_8}{(10)(9)} = \frac{-7\cdot 5\cdot 3\cdot c_0}{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

For $k=9$

$$c_{11} = \frac{8c_9}{(11)(10)} = 0$$

For $k=10$

$$c_{12} = \frac{9c_{10}}{(12)(11)} = \frac{-9\cdot 7\cdot 5\cdot 3\cdot c_0}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}$$

$$c_3 = c_5 = c_7 = c_9 = c_{11} = 0$$

$$c_{2k+1} = 0, \quad k = 1, 2, \dots$$

$$c_2 = \frac{-c_0}{2}, \quad c_4 = \frac{-c_0}{4\cdot 3\cdot 2}, \quad c_6 = \frac{-3c_0}{6\cdot 5\cdot 4\cdot 3\cdot 2}, \quad c_8 = \frac{-5\cdot 3\cdot c_0}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}, \dots$$

$$c_{2k} = \frac{(-1)3\cdot 5\cdot 7\cdot 9 \dots (2k-3)c_0}{(2k)!}$$

$$= \frac{(-1)2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9 \dots (2n-2)(2n-2)(2n-1)(2n)\, c_0}{2\cdot 4\cdot 6\cdot 8 \dots 2n(2n-1)(2n)!}$$

$$= \frac{(-1)(2k)!}{2^n k!(2k-1)(2k)!} c_0$$

$$= \frac{-c_0}{2^n n!(2n-1)}$$

$$c_{2k} = \frac{(-1)\, 3\cdot 5\cdot 7\cdot 9 \dots (2k-3)\, c_0}{(2k)!}$$

$$= \frac{(-1)\, 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9 \dots (2k-3)(2k-2)(2k-1)(2k)\, c_0}{2\cdot 4\cdot 6\cdot 8 \dots 2n(2n-1)(2n)!}$$

$$= \frac{(-1)\,(2k)!\, c_0}{2^k k!\,(2k-1)(2k)!}$$

$$= -\frac{c_0}{2^k k!(2k-1)}$$

From eqn (2)

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$$

$$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + c_{11} x^{11} + c_{12} x^{12} + \dots$$

$$= c_0 + c_1 x + c_2 x^2 + c_4 x^4 + c_6 x^6 + c_8 x^8 + c_{10} x^{10} + c_{12} x^{12} + \dots$$

$$(\because c_3 = c_5 = c_7 = c_9 = c_{11} = 0)$$

$$= c_0 + \sum_{k=1}^{\infty} c_{2k} x^{2k} + c_1 x$$

$$= c_0 + \sum_{k=1}^{\infty} \frac{(-1)\, c_0}{2^k k!(2k-1)} x^{2k} + c_1 x$$

$$= c_0 \left[1 - \sum_{k=1}^{\infty} \frac{x^{2k}}{2^k k!(2k-1)} \right] + c_1 x$$

$$\therefore \phi(x) = c_0 \phi_1(x) + c_1 \phi_2(x)$$

where, $\phi_1(x) = 1 - \sum_{k=1}^{\infty} \frac{x^{2k}}{2^k k!(2k-1)}$

$$\phi_2(x) = x$$

~~Suppose, $\phi_1(x) = 1 -$~~ $\sum_{k=1}^{\infty} \frac{x^{2k}}{2^k k!(2k-1)}$

~~where, $d_k = x$~~

Suppose $\phi_1(x) = 1 + \sum_{k=1}^{\infty} d_k$

where, $d_k = \dfrac{x^{2k}}{2^k k!(2k-1)}$

$$\lim_{k\to\infty}\left|\frac{d_{k+1}}{d_k}\right| = \lim_{k\to\infty}\left|\frac{\dfrac{x^{2k+2}}{2^{k+1}(k+1)!(2k+2-1)}}{\dfrac{x^{2k}}{2^k k!(2k-1)}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{x^{2k+2}}{\underbrace{2^{k+1}(k+1)!}_{2^k\cdot 2(k+1)k!}(2k+1)} \times \frac{2^k k!(2k-1)}{x^{2k}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{x^2(2k-1)}{2(k+1)(2k+1)}\right|$$

$$\therefore \lim_{k\to\infty}\left|\frac{d_{k+1}}{d_k}\right| = 0 < 1$$

This implies $\phi_1(x)$ has convergent power series expansion

Similarly we can show that the function $\phi_2(x)$ has convergent power series expansion for $|x| < \infty$

2) Given

$$y'' + 3x^2y' - xy = 0 \quad \text{——} \ (1)$$

Here, $a_1(x) = 3x^2$, $a_2(x) = -x$ are all analytic for x_0

The solution are also analytic at the ordinary point $x_0 = 0$

$$\phi(x) = \sum_{k=0}^{\infty} C_k (x-x_0)^k$$

$$\phi(x) = \sum_{k=0}^{\infty} C_k x^k \quad \text{——} \ (2)$$

be a power series solution of eq^n (1)

i.e. $\phi''(x) + 3x^2\phi'(x) - x\phi(x) = 0 \quad \text{——} \ *$

we have

$$\phi(x) = \sum_{k=0}^{\infty} C_k x^k$$

$$\phi'(x) = \sum_{k=1}^{\infty} C_k k x^{k-1}$$

$$= \sum_{k=0}^{\infty} C_{k+1}(k+1)x^k$$

$$\phi''(x) = \sum_{k=2}^{\infty} C_k k(k-1)x^{k-2}$$

$$= \sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k$$

Substituting these values in eq^n *.

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k + 3x^2\sum_{k=0}^{\infty} C_{k+1}(k+1)x^k - x\sum_{k=0}^{\infty} C_k x^k = 0$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k + 3\sum_{k=0}^{\infty} C_{k+1}(k+1)x^{k+2} - \sum_{k=0}^{\infty} C_k x^{k+1} = 0$$

$$\sum_{k=0}^{\infty} C_{k+2}(k+2)(k+1)x^k + 3\sum_{k=2}^{\infty} C_{k-1}(k-1)x^k - \sum_{k=1}^{\infty} C_{k-1} x^k = 0$$

$$2C_2 + C_3(3)(2)\cdot x + \sum_{k=2}^{\infty} C_{k+2}(k+2)(k+1)x^k + 3\sum_{k=2}^{\infty} C_{k-1}(k-1)x^k - C_0 x - \sum_{k=2}^{\infty} C_{k-1} x^k = 0$$

$$2C_2 + (6C_3 - C_0)x + \sum_{k=2}^{\infty}\Big[C_{k+2}(k+2)(k+1) + C_{k-1}\big(3(k-1)-1\big)\Big]x^k = 0$$

$$2C_2 + (6C_3 - C_0)x + \sum_{k=2}^{\infty}\left[C_{k+2}(k+2)(k+1) + C_{k-1}(3k-4)\right]x^k$$

$2C_2 = 0 \Rightarrow C_2 = 0 \qquad 2\cdot 3C_3 - C_0 = 0 \Rightarrow C_3 = \frac{C_0}{2\cdot 3}$

$$C_{k+2}(k+2)(k+1) + C_{k-1}(3k-4) = 0$$

$$C_{k+2} = -\frac{(3k-4)\,C_{k-1}}{(k+2)(k+1)} \qquad k = 2, 3, \ldots$$

This eqⁿ is called recurrence relation.

$k = 2, 3, \ldots$

$k = 2$

$$C_4 = \frac{-2C_1}{4\cdot 3}$$

$k = 3$

$$C_5 = \frac{-5C_2}{5\cdot 4} = 0$$

$k = 4$

$$C_6 = \frac{-8C_3}{6\cdot 5} = \frac{-8\cdot C_0}{2\cdot 3\cdot 5\cdot 6}$$

$k = 5$

$$C_7 = \frac{-11C_4}{7\cdot 6} = \frac{2\cdot 11\cdot C_1}{3\cdot 4\cdot 6\cdot 7}$$

$k = 6$

$$C_8 = \frac{-14C_5}{8\cdot 7} = 0$$

$k = 7$

$$C_9 = \frac{-17C_6}{9\cdot 8} = \frac{17\cdot 8\cdot C_0}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}$$

$k = 9$

$$C_{10} = \frac{-20C_7}{10\cdot 9} = \frac{-20\cdot 11\cdot 2\ C_1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}$$

$C_{3k+2} = 0$

$$C_{3k+1} = \frac{(-1)^k C_1\ 2\cdot 11 \ldots (3k-4)}{(3k+1)(3k) \ldots 7\cdot 6\cdot 4\cdot 3}$$

$$C_{3k} = \frac{(-1)^k C_0\cdot 8\cdot 17 \ldots (3k-4)}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9 \ldots (3k-1)(3k)}$$

4) Given $y'' - xy = 0$ —— ①

Here, $a_1(x) = 0$, $a_2(x) = x$

are all analytic for x_0

The solution are also analytic at the ordinary point $x_0 = 0$

$$\phi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k$$

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$$

be a power series solution of eqⁿ ①

i.e. $\phi''(x) - x\phi(x) = 0$ —— $*$

we have

$$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$$

$$\phi'(x) = \sum_{k=1}^{\infty} c_k k x^{k-1}$$

$$= \sum_{k=0}^{\infty} c_{k+1} (k+1) x^k$$

$$\phi''(x) = \sum_{k=2}^{\infty} c_k k(k-1) x^{k-2}$$

$$= \sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^k$$

substituting this values in eqⁿ $*$

$$\sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^k - x \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^k - \sum_{k=0}^{\infty} c_k x^{k+1} = 0$$

$$\sum_{k=0}^{\infty} c_{k+2} (k+2)(k+1) x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty} c_{k+2} (k+2)(k+1) x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty} \left[c_{k+2} (k+2)(k+1) - c_{k-1} \right] x^k = 0$$

$2c_2 = 0 \Rightarrow c_2 = 0$

$c_{k+2}(k+2)(k+1) - c_{k-1} = 0$

$c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)} \qquad k = 1, 2, \ldots$

This eq^n is called recurrence relation

by given $k = 1, 2, \ldots$

we obtain the constant

For $k=1$

$c_3 = \frac{c_0}{3 \cdot 2}$

For $k=2$

$c_4 = \frac{c_1}{4 \cdot 3}$

For $k=3$

$c_5 = \frac{c_2}{5 \cdot 4} = 0$

For $k=4$

$c_6 = \frac{c_3}{6 \cdot 5} = \frac{c_0}{6 \cdot 5 \cdot 3 \cdot 2}$

For $k=5$

$c_7 = \frac{c_4}{7 \cdot 6} = \frac{c_1}{7 \cdot 6 \cdot 4 \cdot 3}$

For $k=6$

$c_8 = \frac{c_5}{8 \cdot 7} = 0$

For $k=7$

$c_9 = \frac{c_6}{9 \cdot 8} = \frac{c_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}$

For $k=8$

$c_{10} = \frac{c_7}{10 \cdot 9} = \frac{c_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}$

For $k=9$

$c_{11} = \frac{c_8}{11 \cdot 10} = 0$

$c_{3k+2} = 0$

$c_{3k} = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \ldots (3k-1)(3k)}$

$c_{3k+1} = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \ldots 3k(3k+1)}$

From eq^n ②,

$\phi(x) = \sum_{k=0}^{\infty} c_k x^k$

$= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \ldots$

$= c_0 + (c_3 x^3 + c_6 x^6 + c_9 x^9 + \ldots) + c_1 x + (c_4 x^4 + c_7 x^7 + c_{10} x^{10} + \ldots)$

$(\because c_2 = c_5 = c_8 = c_{11} = 0)$

$= c_0 + \sum_{k=1}^{\infty} c_{3k} x^{3k} + c_1 x + \sum_{k=1}^{\infty} c_{3k+1}$

$= c_0 \left\{ + \sum_{k=1}^{\infty} \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \ldots (3k-1)(3k)} x^{3k} \right.$

$+ c_1 x + \sum_{k=1}^{\infty} \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \ldots 3k(3k+1)} x^{3k+1}$

$= c_0 + \left[1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \ldots (3k-1)(3k)} \right]$

$+ c_1 \left[x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \ldots (3k)(3k+1)} \right]$

$\therefore \phi(x) = c_0 \phi_1 + c_1 \phi_2(x)$

where, $\phi_1(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \ldots (3k-1)(3k)}$

$\phi_2(x) = x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10 \ldots (3k)(3k+1)}$

* Legendary polynomial :-

The polynomial of $P_n(x)$ of degree n.

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

where n is constant.

$P_n(1)=1$ is called n^{th} ~~linear~~ legendary polynomial.

how

IMP Que- Show that the function P_n is given by

$P_n(x) = \frac{1}{2^n . n!} \frac{d^n}{dx^n}(x^2-1)^n$ — ① is the n^{th} legendary polynomial.

Let ϕ be a polynomial of degree n define by

$\phi(x) = \frac{d^n}{dx^n}(x^2-1)^n$ — **

Let $u(x) = (x^2-1)^n$ — *

Then, $u'(x) = n(x^2-1)^{n-1} 2x$

$u'(x) - n(x^2-1)^{n-1} 2x = 0$

Multiply by x^2-1

$(x^2-1)u'(x) - 2nx(x^2-1)^n = 0$

$(x^2-1)u'(x) - 2nx\,u(x) = 0$ — by *

$u = u(x)$

$(x^2-1)u' - 2nxu = 0$ — ①

First differentiate of eqn ① gives

$(x^2-1)u'' + u'.2x - 2nxu' - 2nu = 0$

$(x^2-1)u'' + (1-n)2xu' - 2nu = 0$

second differentiate of eqn ① gives

$(x^2-1)u''' + u''(2x) + (1-n)2xu'' + 2(1-n)u' - 2nu' = 0$

$(x^2-1)u''' + (1+1-n)2xu'' + 2[(1-n)-n]u' = 0$ — **

$(x^2-1)u''' + (2-n)2xu'' + 2(1-2n)u' = 0$

i.e. from eq^n **

$(x^2-1)u''' + (2-n)2x\cdot u'' - 2[(n-1)+(n-0)]u' = 0$

Now, the third differentiate eq^n ①

$(x^2-1)u^{iv} + u'''(2x) + (2\cdot 3-n)2x\cdot u''' + u''(2-n)2$
$-2[(n-1)+(n-0)]u'' = 0$

$\Rightarrow (x^2-1)u^{iv} + 2xu'''(1+2-n) + u''(2-n)$
$2u''[(2-n)-(n-1)-(n-0)] = 0$

$(x^2-1)u^{iv} + 2xu'''(3-n) - 2[(n-2)+(n-1)+(n-0)]$
$u''' = 0$

⋮

$(n+1)^{th}$ differentiating eq^n ①.

$(x^2-1)u^{n+2} + 2x(n+1-n)u^{n+1} - 2[n-n+(n-(n-1))+(n-(n-2)) \ldots$
$\ldots (n-2)(n-1)(n-0)] = 0$

$(x^2-1)u^{n+2} + 2xu^{n+1} - n(n+1)u^n = 0$

$\Rightarrow (x^2-1)u^{n+2} + 2xu^{n+1} - 2\left[\frac{n(n+1)}{2}\right]u^n = 0$

i.e. $(1-x^2)u^{n+2} - 2xu^{n+1} + n(n+1)u^n = 0$

$\therefore (1-x^2)\phi'' - 2x\phi' + n(n+1)\phi = 0$ $(u^n = \phi)$
$u^{n+1} = \phi'$, $u^{n+2} = \phi''$

$\therefore \phi$ be the solution of Legendry eq^n.

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Now, $\phi(x) = \frac{d^n}{dx^n}(x^2-1)^n$ $(a^2-b^2) = (a-b)(a+b)$

$= \frac{d^n}{dx^n}\left[(x-1)^n \cdot (x+1)^n\right]$

$\frac{d}{dx}(f\cdot g) = (D^n f)g + {}^nC_1(D^{n-1}f)(Dg) + \ldots + f(D^n g)$

$\therefore \phi(x) = \left[\frac{d^n}{dx^n}(x-1)^n\right](x+1)^n + {}^nC_1\left[\frac{d^{n-1}}{dx^{n-1}}(x-1)^n\right]\left[\frac{d}{dx}(x+1)^n\right]$
$+ \ldots + {}^nC_n (x-1)^n\left[\frac{d^n}{dx^n}(x+1)^n\right]$ —②

Suppose $f = (x-1)^n$

$\therefore f' = n(x-1)^{n-1}$

$f'' = n(n-1)(x-1)^{n-2}$

$f''' = n(n-1)(n-2)(x-1)^{n-3}$

⋮

$f^n = n(n-1)(n-2)\ldots 4\cdot 3\cdot 2\cdot (x-1)^{n-n}$ how
$= n!$

Hence eq^n ② become

$\phi(x) = n!(x+1)^n +$ terms containing $(x-1)$

$\phi(x) = n!(x+1)^n$ (if we put $x=1$ then all values are variables i.e. 0)

$\therefore \phi(1) = n!\,2^n + 0$

$\phi(1) = n!\,2^n$

Define $P_n(x) = \frac{1}{2^n n!}\phi(x)$ from ①

clearly,
$P_n(x)$ be the polynomial sol^n of legendre eq^n.

$P_n(x) = \frac{1}{2^n n!}\phi(x)$

$P_n(1) = \frac{1}{2^n n!}\phi(1)$

$= \frac{1}{2^n n!}2^n n!$

$P_n(1) = 1$

Thus $P_n(x)$ is legendre polynomial.

IMP
[Ex. 2] Show that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$ then
MCQ

$P_n(-x) = (-1)^n P_n(x)$

& $P_n(-1) = (-1)^n$

The legendre poly is given by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$$

$$= \frac{1}{2^n n!} D^n[(x^2-1)]^n$$

$$= \frac{1}{2^n n!} \cdot D^n[(x-1)^n(x+1)^n]$$

$$= \frac{1}{2^n n!}[D^n((x-1)^n)(x+1)^n + {}^nC_1 D^{n-1}((x-1)^n) \cdot D(x+1)^n + \dots + {}^nC_n (x-1)^n D^n((x+1)^n)] \quad \text{--- (1)}$$

Now, $D^n(x-1)^n = n!$

$D^{n-1}(x-1)^n = n!(x-1)$

$\therefore$ eqⁿ (1) become

$$P_n(x) = \frac{1}{2^n n!}[n!(x+1)^n + n!(x-1)n(x+1)^{n-1} \cdot n + \dots + (x-1)^n n!] \quad \text{--- (2)}$$

If we put $x=-1$ in eqⁿ (2), we get

$$\therefore P_n(-1) = \frac{1}{2^n n!}(-2)^n n!$$

$$= \frac{1}{2^n n!}(-1)^n(2)^n n!$$

$$P_n(-1) = (-1)^n$$

Similarly, put $x=1$ in eqⁿ (2) we get

$$P_n(1) = \frac{1}{2^n n!}[2^n n!]$$

$$P_n(1) = 1$$

Now, replacing x by $-x$ in eqⁿ (2), we get

$$P_n(-x) = \frac{1}{2^n n!}[\cancel{n!}(-x+1)^n n! + [n!(x-1)^n n(x+1)^{n-1} n + \dots + (-x-1)^n n!]$$

$$= \frac{1}{2^n n!}[(-1)^n[(x-1)^n n! + (n!(x+1)n(x-1)^{n-1} \cdot n + (x+1)^n n!]$$

$$P_n(x) = \frac{1}{2^n n!}(-1)^n[(x+1)^n n! + \dots + (x-1)^n n!]$$

$$P_n(x) = \frac{x}{2^n n!}(-1)^n P_n(x) \quad (\because \text{by (2)})$$

$$\therefore P_n(-x) = (-1)^n P_n(x)$$

Hence, the proof.

[Ex. 3] Show that $\int_{-1}^{1} P_n(x) P_m(x)\,dx = 0$ if $n \neq m$

we know the legendre poly. $P_n(x)$ is satisfies the eqⁿ

$$P_n(x) = \frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}(x^2-1)^n$$

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \quad \text{--- (1)}$$

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \quad \text{--- (2)}$$

($\because P_n(x)$ satisfies the Legendre eqⁿ)

Similarly

$P_m(x)$ also satisfy the eqⁿ (1).

$$(1-x^2)P_m''(x) - 2xP_m'(x) + m(m+1)P_m(x) = 0 \quad \text{--- (3)}$$

Now, multiply eqⁿ (2) by $P_m(x)$ & (3) by $P_n(x)$ & substracting we get

$$(1-x^2)P_n''(x)P_m(x) - 2xP_n'(x)P_m(x) + n(n+1)P_n(x)P_m(x) = 0$$

$$-(1-x^2)P_m''(x)P_n(x) \mp 2xP_m'(x)P_n(x) \pm m(m+1)P_m(x)P_n(x) = 0$$

$$(1-x^2)[P_n''(x)P_m(x) - P_m''(x)P_n(x)] - 2x[P_n'(x)P_m(x) - P_m'(x)P_n(x)] + [n(n+1) - m(m+1)]P_n(x)P_m(x)$$

$$\Rightarrow \frac{d}{dx}[(1-x^2)(P_n'(x)P_m(x) - P_n(x)P_m'(x))] =$$

$$= [m(m+1) - n(n+1)]P_n(x)P_m(x)$$

Integrating betⁿ the limit -1 to 1.

$$\therefore \int_{-1}^{1} 0 = [m(m+1) - n(n+1)]\int_{-1}^{1} P_n(x)P_m(x)\,dx$$

$$[m(m+1) - n(n+1)]\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0.$$

$\therefore \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad (m \neq n)$

Hence, the proof.

IMP Ex. Show that $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$ is a solution of the eq^n $y'' - 2xy' + 2ny = 0$ (where, n is constant)

Let $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$

Here, $u = e^{-x^2}$ —— ①

Differentiating w.r.t. x, we get

$\frac{du}{dx} = -2xe^{-x^2}$

$\frac{du}{dx} + 2xe^{-x^2} = 0$

Differentiating we get n times.

$\frac{d^{n+1}u}{dx^{n+1}} + \frac{d^n}{dx^n}[2xe^{-x^2}] = 0$

$\frac{d^{n+1}u}{dx^{n+1}} + 2x\frac{d^n}{dx^n}(u(x)) + 2n\frac{d^{n-1}}{dx^{n-1}}(u(x)) + 0 = 0$ —— ②

$(f.g)^n = f^n.g + {}^nC_1 f^{n-1}g' + {}^nC_2 f^{n-2}g'' + \dots + {}^nC_n f g^n$

Multiply eq^n ② by $(-1)^n e^{x^2}$

$(-1)^n e^{x^2}\frac{d^{n+1}u}{dx^{n+1}} + (-1)^n e^{x^2} 2x \frac{d^n}{dx^n}(u(x)) + (-1)^n e^{x^2} 2n \frac{d^{n-1}}{dx^{n-1}}(u(x)) = 0$ —— ③

It is given that

$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$ —— *

Replace n by n+1, we get

$H_{n+1}(x) = (-1)^{n+1} e^{x^2}\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2})$ $\quad e^{-x^2} = u$

$-H_{n+1}(x) + 2xH_n(x) - 2nH_{n-1}(x) = 0$

i.e. $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$ —— ④ $\quad n \geq 1$

Now, diff. eq^n * w.r.t. x, we get

$H_n'(x) = (-1)^n 2xe^{x^2}\cdot\frac{d^n}{dx^n}(e^{-x^2}) + (-1)^n e^{x^2}\frac{d^{n+1}}{dx^{n+1}} e^{-x^2}$

$H_n'(x) = 2xH_n(x) - H_{n+1}(x)$ —— ⑤

Using eq^n ④ in eq^n ⑤ we get

$H_n'(x) = 2nH_{n-1}(x)$ —— ⑥

Diff. eq^n ⑤ w.r.t. x we get

$H_n''(x) = 2H_n(x) + 2xH_n'(x) - H_{n+1}'(x)$ — ⑦

Using ⑥ for n = n+1

$H_{n+1}'(x) = 2(n+1)H_n(x)$

from eq^n ⑦

$H_n''(x) = 2H_n(x) + 2xH_n'(x) - 2(n+1)H_n(x)$

$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$

This shows that the $H_n(x)$ is the sol^n of the eq^n $y'' - 2xy' + 2ny = 0$

Hence, the proof.

Linear eq^n with regular singular point :-

Singular point :-

If $x = 0$ is infinite then it is singular point

Ordinary point

$x = 1, 2, \dots \rightarrow$ Finite

Other than singular

Regular singular point.

$x_0 = 0 \qquad P(x)(x-x_0)$

$\qquad\qquad Q(x)(x-x_0)^2$